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Hamiltonian and Pseudo-Hamiltonian Cycles and Fillings In Simplicial Complexes

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Abstract

We introduce and study a d-dimensional generalization of Hamiltonian cycles in graphs - the Hamiltonian d-cycles in K_n^d (the complete simplicial d-complex over a vertex set of size n). Those are the simple d-cycles of a complete rank, or, equivalently, of size $1 + \binom{n-1}{d}$.

The discussion is restricted to the fields $\mathbb{F}_2$ and $\mathbb{Q}$. For $d = 2$, we characterize the n's for which Hamiltonian 2-cycles exist. For $d = 3$ it is shown that Hamiltonian 3-cycles exist for infinitely many n's. In general, it is shown that there always exist simple d-cycles of size $\binom{n-1}{d} - O(n^{d-3})$. All the above results are constructive.

Our approach naturally extends to (and in fact, involves) d-fillings, generalizing the notion of T-joins in graphs. Given a $(d-1)$-cycle $Z^{d-1} \in K_n^d,$ F is its d-filling if $\partial F = Z^{d-1}$. We call a d-filling Hamiltonian if it is acyclic and of a complete rank, or, equivalently, is of size $\binom{n-1}{d}$. If a Hamiltonian d-cycle Z over $\mathbb{F}_2$ contains a d-simplex σ, then $Z \setminus \sigma$ is a a Hamiltonian d-filling of $\partial\sigma$ (a closely related fact is also true for cycles over $\mathbb{Q}$). Thus, the two notions are closely related.

Most of the above results about Hamiltonian d-cycles hold for Hamiltonian d-fillings as well.

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1 Introduction

Combinatorial topology (more precisely, the Homology theory for simplicial complexes) provides a natural framework allowing to generalize the fundamental graph-theoretic notions such as cycles, trees, cuts, expanders, Laplacians, etc., to $(d+1)$-uniform hypergraphs, viewed as pure d-dimensional simplicial complexes. Historically, this framework was used and developed mostly to serve the needs of other disciplines, first and foremost the Algebraic Topology, and, more recently, e.g., the digital processing of visual data. In recent decades it came under investigation for its own sake, resulting in new beautiful results and applications, see [1, 2, 3, 5] to name but a few.

The key notions studied in this paper are d-cycles and acyclic d-fillings of a maximum possible size. For simplicity consider first the one-dimensional case over the field $\mathbb{F}_2$, i.e., graphs. Given a set E of edges over the vertex set V, define $\partial_1 E$, the boundary of E, as the set of all vertices incident to an odd number of edges in E. The set E is a 1-cycle if $\partial_1(E) = \emptyset$. A set E is called acyclic if it contains no cycles. A maximal acyclic set is called a 1-tree. It is a basic fact that all maximal acyclic sets have the same size, which is $|V| - 1$. It is a simple exercise to show that for any even set of vertices$Z \subset V$ [1], there exists a set of edges F over V with $\partial_1 F = Z$. I.e., it is a graph whose set of odd degree vertices is Z. Such F is classically called a Z-join. In view of higher-dimensional generalization to come, we shall call it a 1-filling of Z. It is easy to verify that there exists a 1-fillings of Z that is acyclic. In a special case when $Z = \{a, b\}$, an acyclic filling of Z of the largest possible size is a Hamiltonian path whose end points are Z. Together with the pair (a, b) it forms a Hamiltonian cycle - the largest possible simple cycle (that is, as cycle that does not contain a proper cycle as a subset).

[1] This is a 0-dim cycle, see Section 1.1.

This can naturally be generalized to higher dimensions: instead of pairs, let T be a set of triplets (sets of size 3) over the set of vertices V. In this case the boundary $\partial_2(T)$ (over $\mathbb{F}_2$) is the set of pairs of vertices, each that is incident to an odd number of triangles. Again, a 2-cycle is a set of triplets with empty boundary, and acyclic sets of triplets are those containing no cycles. A simple cycle is a cycle that does not contain a proper subset that is by itself a cycle. It turns out (from the same algebraic reasoning as for graphs) that all the maximal acyclic sets have the same size, which is $\binom{n-1}{2}$, where $n = |V|$. In addition, any 1-cycle Z over V has an acyclic 2-filling F, i.e., an acyclic set of triplets F with $\partial_2 F = Z$.

How large can a simple 2-cycle over an n-size vertex set be? The two-dimensional case is much less obvious, and to our best knowledge, was not systematically studied so far. The following upper bound is simple: the removal of a triangle form a simple cycle creates an acyclic set. Since all acyclic sets are of size at most $r(n, 2) = \binom{n-1}{2}$, it follows that an absolute upper bound is $r(n, 2) + 1$. Would such a simple 2-cycle exist, it would be called Hamiltonian 2-cycle. Note the connection to fillings: If Z is a simple 2-cycle containing a triangle σ, then $Z \setminus \{\sigma\}$ is a 2-filling of the three pairs that are the boundary of σ (and are a 1-cycle).

For the lower bound on the largest 2-simple cycle, is has been known for some time that there exist simple 2-cycles of size $c_2 \cdot r(n, 2)$ for some constant $0 < c_2 < 1$. E.g., the important Complete Graph Embedding Theorem (implying the tightness of Heawood's bounds on the chromatic number of graphs embeddable in 2-surfaces of a prescribed genus; see e.g., the book [6]) claims that any K_n, $n \geq 4$, $n \equiv 0, 1 \pmod 3$, is (efficiently) realizable as a triangulation of (both orientable, and nonorientable) 2-surface. This gives an explicit construction of a simple 2-cycle of a size $\approx \frac{2}{3} r(n, 2)$.

All the above notions are generalized to higher dimensions. In this case the size of a maximum simple d-cycle on V of size n is at most $r(n, d) + 1$, where $r(n, d) = \binom{n-1}{d}$, due to the rank argument, and at least $c_d \cdot r(n, d)$ for some (small) constant $c_d > 0$. This follows, e.g., from the study of the threshold probabilities for random simplicial d-complexes by Linial et al. [7].

It this paper we completely resolve the two-dimensional case, and (constructively) show that the size of a largest simple 2-cycle is $r(n, 2)$ when $n \equiv 1, 2 \pmod 4$, and $r(n, 2) + 1$ when $n \equiv 0, 3 \pmod 4$. Hence, Hamiltonian 2-cycles exist for the latter case. In dimension 3 we construct Hamiltonian simple 3-cycles, that is of size $r(n, 3)+1$, for an infinite sequence of n's and in general, we construct simple d-dimensional cycles of size $(1 - O(1/n^3)) \cdot r(n, d)$.

Observe that any nontrivial simple d-cycle Z can be represented in a form $Z = \sigma_d - F^{(d)}$, where $\sigma_d \in Z$ is a d-simplex, and $F^{(d)}$ is an acyclic d-filling of $\partial_d \sigma_d$. Thus, constructing large simple d-cycles is equivalent to constructing large acyclic d-fillings of $\partial_d \sigma_d$. It is natural to generalize this question to what is the maximum possible size of an acyclic d-filling $F^{(d)}$ of a (any) given nontrivial $(d-1)$-cycle Z, with respect to set of vertices V, $|V| = n$. The rank argument immediately implies that $|F^{(d)}| \leq r(n, d)$. For $d = 2$ we completely resolve the case and for $d > 2$ we construct an acyclic d-filling of size $(1 - O(1/n^3)) \cdot r(n, d)$ for any nontrivial $(d-1)$-cycle Z.

We end with a remark that while the basic definition of boundary was defined above with respect to $\mathbb{F}_2$, all notions and results extend also to boundaries with respect to $\mathbb{Q}$, or any other field.

Finally, a note about the methods: The paper is combinatorial in nature. Its use of Homology theory does not go beyond the basic definitions, and the basic properties of the resulting structures. This is partially due to a systematic use of a very special type of acyclic sets of d-simplices, and the d-chains supported on them. Such sets, defined in a purely combinatorial manner by means of a certain conical extension (see Claim 1.1 below), are quite tractable by combinatorial means, and may prove useful for future studies.

1.1 Terminology and Preliminaries Pertaining to Simplicial Complexes

1.1.1 Basic Standard Notations

The notation $[n]$ is a shorthand for the set $\{1, \ldots, n\}$. If A and B are sets, then $A \oplus B$ denotes their symmetric difference; if A and B are vectors over $\mathbb{F}_2$, then it denotes their vector sum.

simplices and Complexes. An abstract *d-dimensional simplex* (or *d-simplex* for short) can be identified with a set of size $d+1$. An abstract *simplicial complex* X is a collection of simplices that is closed under containment. In this case, the simplices in X are also called *faces*. The set of all the 0-simplices in X is called the *vertex-set* $V(X)$ of X. In this paper we shall always assume that $V(X)$ is finite and often identify it with $[n]$, where $n = |V(X)|$. The *dimension* of a simplex is the size of its vertex set minus 1. The *dimension* of a simplicial complex X is the maximum dimension of a simplex in X. Further, X is called *pure* if all its maximal faces are of the same dimension.

The set of all i-dimensional simplices of X, the i-*skeleton* of X, is denoted by $X^{(i)}$.

The *complete* d-dimensional simplicial complex on $[n]$, $K_n^d = \{\sigma \subset [n] : |\sigma| \leq d+1\}$, contains all the simplices on $[n]$ of dimension $\leq d$.

The *degree* of a k-face σ in a *pure* d-dim simplicial complex X, denoted $deg(\sigma, X)$, is the number of d-faces in X which contain σ.

Orientations, Chains, and the Boundary Operator. An *orientation* of a simplex is the equivalence relation on all the permutations on $V(\sigma)$, that is - orderings of the vertices, in which two permutations are equivalent if one being an even permutation of the other. Hence, there are two possible orientations of a d-simplex of dimension ≥ 2, and one orientation for $d < 2$. An *oriented simplex* is a simplex with orientation. An oriented simplicial complex is a simplicial complex whose simplices are oriented.

Given a field $\mathbb{F}$ and an oriented simplicial complex X, an $\mathbb{F}$-weighted formal sum C of the (oriented) k-faces of X is called a k-*chain* on X over $\mathbb{F}$, i.e., $C = \sum_{\sigma \in X^{(k)}} c_\sigma \sigma$, where $c_\sigma \in \mathbb{F}$. All different orderings of a d-simplex are divided to two equivalent classes, represented by the $\{-1, +1\}$ signs. Over $\mathbb{F}_2$ the notion of a sign is vacuous. The importance of the signs is when considering the boundary operator, to be discussed below.

The support $supp(C)$ of k-chain C is the set of non-oriented k-simplices σ such that $c_\sigma \neq 0$. The *size* of C is defined as $|C| = |supp(C)|$. The collection of all d-chains on K_n^d form a vector space $\mathcal{C}_d$ of dimension $\binom{n}{d+1}$. The vertex set of a chain C over K_n^d is $V(C) = V(supp(C))$.

The *boundary* $\partial_d \sigma$ of an oriented d-simplex $\sigma = \{v_0, \ldots, v_d\}$, with $v_0 < \cdots < v_d$, is the $(d-1)$-chain $\sum_{i=0}^{d} (-1)^i \sigma_i$, where $\sigma_i = (\sigma \setminus \{v_i\})$ is the oriented simplex obtained by erasing v_i from the oriented σ as above. The boundary operator is well defined in the sense that it does not depend on the particular orderings (up to corresponding equivalences) chosen to represent σ and σ_i's respectively. Note that $\tau = \sigma - \{v\}$ has a sign above depending on the relative order of v in σ. We denote this sign by $[\sigma : \tau]$. Hence $\partial \sigma = \sum_{v \in \sigma} [\sigma : (\sigma \setminus \{v\}] \cdot \sigma$. The linear extension of this operator to the whole of $\mathcal{C}_d$ is the *boundary operator* $\partial_d : \mathcal{C}_d \longrightarrow \mathcal{C}_{d-1}$. A fundamental property of the boundary operator is $\partial_{d-1} \partial_d = 0$.

When the value of d is unambiguous from the context, the subscript d of ∂_d may be dropped.

Cycles. A d-chain Z is called a d-*cycle* if $\partial_d Z = 0$. We refer to $0 \in \mathcal{C}_d$ as the *trivial* d-*cycle* or the *zero cycle*. Further, when Z is the only nontrivial d-cycle supported on a $supp(Z)$, Z is called *simple*. The collection of all d-cycles of K_n^d form a vector space $\mathcal{Z}_d$ of dimension $\binom{n-1}{d+1}$ over $\mathbb{F}$. Note that for $(d+1)$-simplex σ, $\partial \sigma$ is a non-trivial d-cycle. This is the non-trivial cycle of minimum possible size (for any dimension). It can be verified that the space of d-cycles $\mathcal{Z}_d$ is spanned by $\{\partial_{d+1} \sigma : \sigma \in K_n^{d+1}\}$.

Forests and Hypertrees. A pure d-complex F is called *acyclic* if there no nontrivial d-cycle whose support is a subset of $supp(F)$. Slightly deviating from the standard notation, we shall call such set of d-simplices F a d-*forest*, and, further, call it a d-*hypertree* on $[n]$ if it is a maximal d-forest in K_n^d. Matroid-theoretic considerations immediately imply that all d-hypertrees on $[n]$ have the same size. Consider the d-star in K_n^d, i.e., the set of all d-simplices that contain a fixed vertex v. One can easy verify that it is a maximal forest, i.e., a d-hypertree. Hence, the size of any d-hypertree of K_n^d is equal to the size of d-star, being $\binom{n-1}{d}$.

The set of all $(d-1)$-chains $\{\partial_d \sigma : \sigma \in F\} \subset \mathcal{Z}_{d-1}$ is linearly independent when F is a d-forest, and, moreover, it is a basis of $\mathcal{Z}_{d-1}$ when F is a d-hypertree. This is the *spanning property* of d-hypertrees. In particular, for such F every d-simplex $\sigma \in K_n^d \setminus supp(F)$ defines the *fundamental* d-*cycle of* σ *with respect to* T, being the support of the unique non-trivial d-cycle supported on the $F \cup \{\sigma\}$.

Hypercuts. d-hypercuts of K_n^d are its d-cocycles (equivalently, d-coboundaries) of a minimal support. To avoid the unnecessary discussion of d-cochains and d-cocycles, for the needs of this paper it suffices to say that the supports S of d-hypercuts are precisely the sets of d-simplices obtainable in the following manner. Start with any d-hypertree T of K_n^d and $\sigma \in T$. Then, set S to be the set of all d-simplices τ such that $T \setminus \{\sigma\} \cup \{\tau\}$ is acyclic. See [9, 8] for more details on d-hypercuts.

Finally, we note that over $\mathbb{F}_2$, d-chains (that is, cycles in this context) can be identified with their support.

1.1.2 Less Common Notions, Operators and Facts

Star and Link. While these operators are usually considered in the context of simplicial complexes, they are well defined for chains as well. Given a d-simplex σ and a vertex v the *star* of σ with respect to v is $St(v, \sigma) = 0$ if $v \notin \sigma$ and σ otherwise. Similarly $Lk(v, \sigma) = [\sigma : (\sigma \setminus \{v\})] \cdot (\sigma \setminus \{v\})$. Both operation are extended linearly to chains.

Note that $Lk(v, \sigma) = \partial\sigma - St(v, \partial\sigma)$.

It follows immediately that a link of a d-cycle Z^d is a $(d-1)$-cycle over $V \setminus \{v\}$ since this is immediate for the cycle $\partial\sigma$ for any $(d+1)$-simplex σ, and as commented above these cycles span the space of cycles.

Cone. The *cone* operator is the right inverse of the link operator; For $x \notin \sigma$ it maps a d-simplex σ to the $(d+1)$-simplex $Cone(x, \sigma) = [(\sigma \cup \{v\}) \ : \sigma] \cdot (\sigma \cup \{v\})$. Again, this is linearly extended to any chain C where $x \notin V(C)$.

A simple verification yields:

$$Cone(x, Lk(x, C)) \ = \ St(x, C)\,. \tag{1}$$

and

$$\partial_{d+1} Cone(x, C) \ = \ C - Cone(x, \partial_d C)\,. \tag{2}$$

The following fact about conic extensions is fundamental for this paper. Observe that (with some abuse of notation) the $Cone(x, S)$ operator is well defined not only for d-chains, but also for non-oriented unweighted sets of d-simplices.

Claim 1.1 *Assume that $T^{(d)}$ and $T^{(d-1)}$ are, respectively, a d-forest and a $(d-1)$-forest (a d-hypertree and a $(d-1)$-hypertree) over a field $\mathbb{F}$ and a vertex set V. Then, for $x \notin V$, $T^{(d)} \cup Cone(x, T^{(d-1)})$ is a d-forest (a d-hypertree) over $V \cup \{x\}$.*

Proof. Since $T^{(d)}$ is acyclic and disjoint from $Cone(x, T^{(d-1)})$, any nontrivial d-cycle Z supported on $T^{(d)} \cup Cone(x, T^{(d-1)})$ must contain the the vertex x. Consider $Lk(x, Z)$. On one hand it is a nontrivial $(d-1)$-cycle on V. On the other hand, it is supported on the acyclic $T^{(d-1)}$: Contradiction.

Further, set $|V| = n$. If $T^{(d)}$ and $T^{(d-1)}$ are *hypertrees* over V, they have support of size $\binom{n-1}{d}$, $\binom{n-1}{d-1}$ respectively. Then $T^{(d)} \cup Cone(x, T^{(d-1)})$ has support of size $\binom{n-1}{d} + \binom{n-1}{d-1} = \binom{n}{d}$, and therefore a d-hypertree over $V \cup \{x\}$. ■

A matter of notations

In what follows we often use a superscript d over a chain or a simplicial complex. The superscript denotes the maximal dimension of the corresponding (usually pure) object. Z will always denote a cycle, F or T will denote acyclic chains or sets (that is, forests). Hence e.g., Z^d is a d-cycle.

Fillings. A *filling* of a $(d-1)$-cycle[2] Z^{d-1} over K_n^d is a d-chain $F^{(d)}$ over K_n^d such that $\partial F^{(d)} = Z^{d-1}$. A filling $F^{(d)}$ (and in general, any d-chain) will be called *acyclic* if its support is acyclic. The fact that F is a filling of Z^{d-1} will be denoted as $F = \text{Fill}(Z^{d-1})$.

The *deficit* of an acyclic chain $F^{(d)}$ will be defined as $\text{deficit}(F_n^d) = \binom{n-1}{d} - |F^{(d)}|$. Since $\binom{n-1}{d}$ is the size of every maximal acyclic d-chain in K_n^d, the deficit is never negative.

Let $T \subseteq K_n^d$ be a d-hypertree. For every $(d-1)$-cycle Z^{d-1} on K_n^d there exists a unique acyclic filling of Z^{d-1} supported on T. This immediately follows from the spanning property and the acyclicity of T. In fact, this is a linear bijection between $\mathcal{Z}_{d-1}$, the set of $(d-1)$-cycles of K_n^d, and $\mathcal{C}_d(T)$, the set of d-chains supported on a T.

0-deficit fillings, Hamiltonicity and cycles. A a 0-deficit acyclic filling $F^{(d)}$ of Z^{d-1} in K_n^d is obviously the largest possible filling (in terms of its support). If $Z^{d-1} = \partial\sigma$ for some $\sigma \in K_n^d$, a 0-deficit acyclic filling $F^{(d)}$ of

[2]Formally, fillings should be defined for $(d-1)$-boundaries rather than for $(d-1)$-cycles. However, for K_n^d, as well as for any homologically d-connected complex, the two are the same.

$\partial\sigma$ will be called *Hamiltonian* as $F - \sigma$ is a *simple* cycle of the maximum possible support, namely $\binom{n-1}{d} + 1$. In turn, a simple d-cycle Z^d in K_n^d will be called Hamiltonian if its size is $\binom{n-1}{d} + 1$. Observe that Z^d is Hamiltonian if and only if for any term $c_\sigma\sigma$ in it (where σ is a d-simplex), $Z^d - c_\sigma\sigma$ is an acyclic 0-deficit filling of $\partial\sigma$.

While for graphs Hamiltonian cycles always exist (for any $n \geq 3$), this is not necessarily true for higher dimensional full-simplicial complexes.

2 Large Acyclic d-Dimensional Fillings

Can one expect that every $(d-1)$-cycle Z^{d-1} on K_n^d has a 0-deficit filling? In particular, is there a Hamiltonian d-cycle for every d for large enough n? The answer may depend on the underlying field. For $\mathbb{F}_2$ there is an obvious obstacle for fillings of $(d-1)$-cycles, for even d. Observe that in this case $\partial_d F^{(d)} = Z^{d-1}$ implies that the sum of coefficients (mod 2) of the chain $F^{(d)}$ is equal to that of Z^{d-1}. In other words, the parities of $|F^{(d)}|$ and $|Z^{d-1}|$ must be equal. We call this obstacle 'the parity condition', and it is defined formally below.

Thus if Z^{d-1} has a 0-deficit filling the following parity condition holds.

Definition 2.1 (parity condition) *We say that a non-empty $(d-1)$-cycle over $\mathbb{F}_2$ has the parity condition if d is even and*

$$|Z^{d-1}| \equiv \binom{n-1}{d} \pmod 2 \tag{3}$$

For all we presently know, the following rather strong conjecture may well be true:

Conjecture 2.2 *Over $\mathbb{F}_2$, for every $d \geq 0$ there exists a number n_d, such that every non-trivial $(d-1)$-cycle Z^{d-1} on K_n^d with $n \geq n_d$ has a 0-deficit filling if an only if the parity condition holds. More over, for any non-trivial $(d-1)$-cycle, regardless of the parity condition there is an acyclic filling $F^{(d)}$ of Z^{d-1} of deficit 1.*

Over $\mathbb{Q}$, for significantly large n, Z^{d-1} always has a 0-deficit filling on K_n^d.

In what follows we shall establish this conjecture for $d \leq 2$ (over $\mathbb{F}_2$ and over $\mathbb{Q}$).

Theorem 2.3 *Over $\mathbb{F}_2$, every nonzero 1-cycle Z^1 on K_n^2 has an acyclic filling of deficit at most 1. Further, if the parity condition holds it has a 0-deficit acyclic filling.*

Over $\mathbb{Q}$, every nonzero 1-cycle Z^1 has a 0-deficit acyclic filling on K_n^2 for large enough n.

For $d \geq 3$, we prove a weaker statement:

Theorem 2.4 *Using the notations of Conjecture 2.2, there always exists an acyclic filling $F^{(d)}$ of Z^{d-1} (over $\mathbb{F}_2$ and over $\mathbb{Q}$) on K_n^d of deficit $O(n^{d-3})$. In particular, for $d = 3$, the deficit is constant.*

In all cases the following generic recursive construction, FILL() will be employed. Given a nonzero $(d-1)$-cycle Z_n^{d-1} over K_n^d it reduces the problem to constructing a (large) acyclic $(d-1)$-dimensional fillings for a certain $(d-2)$-cycle and an acyclic filling of a $(d-1)$-cycle, but over a smaller underlying set.

A matter of notations

In what follows the universe over which all simplicial complexes are considered is $V = [n]$. All chains in what follows are pure and are denoted using a subscript and a superscript. The superscript denotes the maximal dimension while the subscript denotes the size of the subset of the universe on which the chain is defined over. The the actual subset of vertices will be either clear from the context, or explicitly defined.

In the recursion below we initially have our universe $V = [n]$. However, during the recursive procedure we choose a special vertex $v_n \in V$. This will define a re-enumeration of V along every recursion path according to this order in which the vertices are chosen. Once v_n is chosen, some next objects over $V \setminus \{v_n\}$ are (recursively) constructed and hence their subscripts will correspondingly be $(n-1)$.

FILL(Z_n^{d-1}, V)   ; *the input* Z_n^{d-1} *is a* $(d-1)$*-cycle over the universe* V.
   ; *The result is an acyclic filling* F_n^d *of* Z_n^{d-1}.
If $Z_n^{d-1} = 0$ *return* 0 *(the zero cycle* $\leftrightarrow$ *empty filling).*
if $d = 0$, *and* $Z_n^{-1} = c \cdot \emptyset$, *return a (suitably chosen) vertex* $v \in V$ *with coefficient* c;

if $d > 0$,
  pick a (suitably chosen) pivot vertex v_n *in* $V(Z_n^{d-1})$;
   $Z_{n-1}^{d-2} \leftarrow Lk(v_n, Z_n^{d-1})$;
   $F_{n-1}^{d-1} \leftarrow \text{FILL}(Z_{n-1}^{d-2}, V \setminus \{v_n\})$;
   $Z_{n-1}^{d-1} \leftarrow Z_n^{d-1} - St(v_n, Z_n^{d-1}) + F_{n-1}^{d-1}$;
  return $F_n^d \leftarrow \text{FILL}(Z_{n-1}^{d-1}, V \setminus \{v_n\}) - Cone(v_n, F_{n-1}^{d-1})$

To make the above generic construction explicit, it remains to specify how to choose the pivot vertices, and the choice of the retuned v in the base case of 0-dim filling. We will prove that regardless of this choice, the output is an acyclic filling of Z_n^{d-1}. A good choice of the pivot vertex will guarantee a lagre size filling.

Before presenting a formal proof we start with the analysis of the procedure in the case $d \leq 1$ and $\mathbb{F} = \mathbb{F}_2$, which could also be taken a base case for the inductive proof for $\mathbb{F}_2$ ahead. In this case we replace $+, -$ over $\mathbb{F}$ with the mod two addition $\oplus$. Note also that for any complex $A \subseteq K_n^r$ and $v \in V$, $A - Star(v, A) = A \setminus \{v\} = \{\sigma \in A | \; v \notin \sigma\}$. Note also that FILL() has formally a parameter indicating the underlying set in respect to which the filling is created, and with respect to which the deficit is defined. In what follows we drop this parameter from the recursive call when ever it is clear from the context.

For $d = 0$, the unique (-1)-dim nonzero cycle is $Z_n^{-1} = \emptyset$. In this case for any vertex $v \in V$, the chain $1 \cdot \{v\}$ namely, the singleton v, is acyclic with boundary $\emptyset$.

For $d = 1$, a non zero 0-cycle Z_n^0 is a non-empty even-size subset of V. In this case an acyclic filling of Z_n^0 is a forest $F \subset K_n^1$ whose odd degree vertices is exactly the vertices in Z_n^0. The existence of a 0-deficit filling in this case can be proven directly from simple combinatorial consideration. In particular for $Z_n^0 = \{u, v\}$ this is any path in K_n^1 whose end points are u, v.

Still, let us analyse the procedure for $d = 1$, namely for an even size set $Z_n^0 \subseteq [n]$: let $v = v_n \in V(Z_n^0)$ be the chosen pivot vertex. Then $Lk(v_n, Z_n^{d-1}) = \emptyset = Z_{n-2}^{-1}$ hence $F_{n-1}^0 = u \in V \setminus \{v\}$. For any such u, $Z_{n-1}^0 = Z_n^0 \oplus \{u, v\}$ is an even set. Either $Z_{n-1} = \emptyset$ (in the case $Z_n^0 = \{v, u\}$) in which case the acyclic filling $Cone(v_n, u) = (v, u)$ is retuned. Otherwise, if $Z_n \neq \{u, v\}$ or $u \notin Z_n^0$ is chosen, Z_{n-1} is a non-empty even subset of $V \setminus v$. In this case a forest F_{n-1}^0 whose odd vertices is returned as $\text{FILL}_{n-1}^0(Z_{n-1}^0, V \setminus \{v\})$ and $F_{n-1}^0 \cup \{(v, u)\}$ is the final answer. Note that by induction (with the right choice of u above, namely $u \neq x$ in the case $Z_n^0 = \{v, x\}$) F_{n-1}^0 being 0-deficit forest is of size $n - 2$ resulting in F_n^0 of size $n - 1$, namely being 0-deficit.

The analysis for $\mathbb{F} = \mathbb{Q}$ is similar and will be skipped.

We end this analysis of the case $d = 1$ with the following claim that will be used later.

Claim 2.5 *For any fixing of* v_n *in the call for* FILL(Z_n^0, V)*, there are at least* $(n-2)!$ *different (labeled)* 0*-deficit* 1*-fillings of (any)* Z_n^0*. In particular, for* $n \geq 4$ *there at least two different fillings.*

Proof. In the case $Z_n^0 = \{v, x\}$ there are $n - 2$ choices of $u \notin Z_n^0$ that form a right choice of u as described above. Each will correspond to a different final F_n^1 as for different u, u' $Cone(v_n, F_{n-1}^{-1})$ contains only (v_n, u) or (v_n, u') respectively. In the case $|Z_n^0| > 2$, u is unrestricted and can take any of the $n - 1$ possible values. Hence the claim follows by induction and the observation that for $n = 3$ there is 1 such filling.

Again, the argument above is made formally for $\mathbb{F}_2$ but a similar argument is done w.r.t $\mathbb{Q}$. ■

Before we prove Theorems 2.3 and 2.4 we first prove that for any field, the procedure returns an acyclic filling.

Lemma 2.6 *Let* Z_n^{d-1} *be any non-zero cycle in* K_n^d*. Procedure* FILL(Z_n^d, V) *returns an acyclic filling* $F_n^d = \text{Fill}(Z_n^{d-1})$ *regardless of the choice of* v_n*. Further,* $\text{deficit}(F_n^d) = \text{deficit}(F_{n-1}^{d-1}) + \text{deficit}(\text{FILL}_{n-1}^{d-1}(Z_{n-1}^{d-1}, V \setminus$

$\{v_n\})$), *where* F^{d-1}_{n-1} *is any acyclic filling* $F^{d-1}_{n-1} = \mathrm{Fill}(Z^{d-2}_{n-1}, V \setminus \{v_n\})$, *and* $Z^{d-2}_{n-1}, Z^{d-1}_{n-1}$ *are the corresponding objects as defined in the procedure.*

Proof. The statement is obviously correct for $d = 0$. Assume inductively that it is correct for all $d' < d$ and for d with $n' < n$.

First, let us verify that Z^{d-1}_{n-1} is a $(d-1)$-cycle as otherwise the procedure is not even well defined. Indeed, since Z^{d-1}_n is a cycle, then $Z^{d-2}_{n-1} = Lk(v_n, Z^{d-1}_n)$ is a cycle as shown is Section 1.1.2. Hence by induction it follows that $\partial F^{d-1}_{n-1} = Z^{d-2}_{n-1}$. In addition, by Equations (1) and (2), $\partial(Star(v_n, Z^{d-1}_n)) = Lk(v_n, Z^{d-1}_n) = Z^{d-2}_{n-1}$. Plugging this into the expression for Z^{d-1}_{n-1} and taking its boundary it follows that

$$\partial_{d-1} Z^{d-1}_{n-1} \;=\; \partial_{d-1} Z^{d-1}_n - Lk(v_n, Z^{d-1}_n) + Z^{d-2}_{n-1} \;=\; 0 - Z^{d-2}_{n-1} + Z^{d-2}_{n-1} = 0$$

Next, we show that F^d_n is a filling of Z^{d-1}_n. Indeed,

$$\partial_d F^d_n \;=\; \partial_d \mathrm{FILL}(Z^{d-1}_{n-1}, V \setminus \{v_n\}) - \partial_d Cone(v_n, F^{d-1}_{n-1}) =$$

$$Z^{d-1}_{n-1} - F^{d-1}_{n-1} + Cone(v_n, \partial F^{d-1}_{n-1}) \;=\; Z^{d-1}_{n-1} - F^{d-1}_{n-1} + St(v_n, Z^{d-1}_{n-1}) \;=\; Z^{d-1}_n$$

where the 2nd equality is by Equation (2), the next is by induction, and the last is by the definition of Z^{d-1}_{n-1}.

It remains to show that F^d_n is acyclic. Again, by induction this holds for F^{d-1}_{n-1} and $F^d_{n-1} = \mathrm{FILL}(Z^{d-1}_{n-1}, V \setminus \{v_n\})$. Hence this directly follows from Claim 1.1.

Finally, $|supp(F^d_n)| = |supp(Cone(v_n, F^{d-1}_{n-1}))| + |supp(\mathrm{FILL}^{d-1}_{n-1}(Z^{d-1}_{n-1}, V \setminus \{v_n\})|$ since these supports are disjoint. Further $|supp(Cone(v_n, F^{d-1}_{n-1}))| = |supp(F^{d-1}_{n-1}))|$, hence, using that fact that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, it follows that $\mathrm{deficit}(F^d_n) = \mathrm{deficit}(F^{d-1}_{n-1}) + \mathrm{deficit}(\mathrm{FILL}^{d-1}_{n-1}(Z^{d-1}_{n-1}, V \setminus \{v_n\}))$ as claimed. ■

2.1 Proof of Conjecture 2.2 for $d = 2$

2.1.1 Filling over $\mathbb{F}_2$

We prove here the following restatement of Theorem 2.3 over $\mathbb{F}_2$.

Theorem 2.7 *Let* $n \geq 4$. *Every nonzero* 1*-cycle* Z^1_n *on* K^2_n *has at least two acyclic fillings of deficit at most* 1 *over* $\mathbb{F}_2$. *Further if the the parity condition holds it has a* 0*-deficit acyclic filling, and for* $n \geq 5$ *it has at least two such fillings.*

Proof. The proof is by induction on n . The case of $n = 3$ is trivial. For $n = 4$, if the cycle is of length 3, the parity condition holds (and there is a unique 0-deficit filling). If the cycle is of length 4 the parity condition does not hold and there are two 1-deficit fillings. For $n = 5$ there are two cycles that meet the parity condition, each has at least two 0-deficit filling. This can be easily checked by the reader.

We assume that the theorem is correct for any $Z^1_{n-1},\ n-1 \geq 5$. Recall that $Z^1_n \;\oplus\; St(v_n, Z^1_n) = Z^1_n \setminus \{v\}$, namely the subgraph obtained from Z^1_n by deleting the vertex v and all simplices that contain it. Assume that $n \geq 6$ and that the parity condition holds for the given Z^1_n. Let $v = v_n \in V(Z^1_n)$ be arbitrary. Then, the procedure FILL sets $Z^1_{n-1} \;=\; (Z^1_n \setminus \{v_n\}) \;\oplus\; F^1_{n-1}$, where $F^1_{n-1} = \mathrm{Fill}(Lk(v_n, Z^1_n),\ V \setminus \{v_n\})$ is a 0-deficit tree in K^1_{n-1}, namely over $V \setminus \{v_n\}$ of size $n-2$. This exists by Claim 2.5, as explained in the preface of this Section.

To complete the construction, namely, to be able to use the induction hypothesis on Z^1_{n-1}, we only need that $Z^1_{n-1} \neq \emptyset$ and that the parity condition is met for it (with $n' = n-1$).

Note that $Z^1_n \setminus \{v_n\} = A$ is fixed and fully determined from Z^1_n once v_n is chosen. Now, for F^1_{n-1} we have $(n-3)! \geq 3$ different legitimate fillings by Claim 2.5. Hence for at least two of them $Z^1_{n-1} = A \oplus F^1_{n-1}$ is not the trivial cycle as needed. Choose one specific such F^1_{n-1}.

Finally, $|Z^1_{n-1}| = |Z^1_n| \oplus |St(v_n, Z^1_n)| \oplus (n-2) \pmod 2$. Note that $|St(v_n, Z^1_n)| \equiv 0(2)$ as Z^1_n is a 1-cycle. It follows that $|Z^1_{n-1}| \equiv |Z^1_n| - (n-2) \equiv \binom{n-1}{2} - (n-2) \equiv \binom{n-2}{2} \pmod 2$. Where the 2nd equality is by the fact that the parity condition holds for Z^1_n. Hence the parity condition holds for Z^1_{n-1}.

To show that there are at least two such fillings, we use the induction on n. Namely, by induction there are at least two 0-deficit fillings $\mathrm{Fill}(Z^1_{n-1})$ for the fixed Z^1_{n-1}. These two fillings result in two distinct fillings in the return statement using the chosen fixed F^1_{n-1}.

For the case that the parity condition does not hold, the same argument as in the last two paragraphs implies that the parity condition does not hold for Z^1_{n-1} too. Hence again by induction we get at least two 1-deficit filling as the deficit of F^1_{n-1} is 0. ∎

2.1.2 $d = 2$ over $\mathbb{Q}$

A analog of Theorem 2.7 for $\mathbb{F} = \mathbb{Q}$ is similar except that there is no parity obstacle. On the other hand, the induction base cases for $n \leq 5$ are different.

Theorem 2.8 *Let $n \geq 4$. For any nontrivial 1-cycle Z^1_n over $\mathbb{Q}$ there exists a 0-deficit 2-filling F^2_n except for the following two cases (the cycles C_i's below are directed, and uniformly weighted).*

$n = 4$ and $Z^1_n = C_4$

$n = 5$ and $Z^1_n = C_3$.

Further, if $n \geq 6$ every 1-cycle has at least two such fillings. In all the exceptional cases there exists 2-fillings of deficit 1.

Proof. Assuming by induction that a 0-deficit filling for $6 \leq n' < n$ exists for every non-trivial 1-cycle $Z^1_{n'}$, the proof for such filling for Z^1_n is immediate and identical to the proof of Theorem 2.7 (with addition over $\mathbb{Q}$ replacing $\oplus$).

For $n \leq 6$ a case analysis is presented in Appendix section A. ∎

3 Proof of Theorem 2.4

Fillings based on procedure FILL are not adequate to proof Conjecture 2.2. The recursive call, even for $d = 3$ uses filling for $d = 2$ in the top level, which may not be 0-deficit due to the parity obstacle in the case of $\mathbb{F}_2$ (which is not an obstacle at all for $d = 3$), or due to the bad base cases for $\mathbb{F} = \mathbb{Q}$.

An application of Theorem 2.8 directly imply a filling for Z^{d-1}_n over $\mathbb{Q}$ of deficit $O(n^{d-3})$, see Section 3.2.

A similar application of Theorem 2.7 would imply a filling for Z^{d-1}_n of deficit $O(n^{d-2})$ over $\mathbb{F}_2$. We aim however for the same bound as for $\mathbb{Q}$. For this we will need to treat the case $d = 3$ more carefully for $\mathbb{F}_2$. This will be done in the following Section 3.1.

3.1 Fillings over $\mathbb{F}_2$

We aim here to prove a slightly stronger results for $d = 3$ and $\mathbb{F}_2$. It asserts that a deficit of at most 1 can always be achieved, and a 0-deficit can also be achieved for a large collection of cycles called *friendly cycles* below.

Recall that for a chain $C \subseteq K^d_n$ and a vertex $u \in [n]$, $deg(u, C) = |St(u, C)|$ namely, it is the number of d-simplices in C that contain u.

Definition 3.1 (friendly cycle) *A cycle Z^2_n is called friendly if there exist two vertices $v', v'' \in V(Z^2_n)$ such that $deg(v', Z^2_n) \not\equiv deg(v'', Z^2_n) \pmod 2$.*

Theorem 3.2 *Let Z^2_n be a friendly 2-cycle over $\mathbb{F}_2$ on K^2_n. Then there exists an acyclic filling F^3_n of Z^2_n of 0-deficit. Moreover, if $n \geq 7$ there are at least 2 such fillings.*

A matter of notations: The recursion call for $\mathrm{FILL}(Z_n^2, V)$ results in a double recursion: one for the lower dimensional $\mathrm{FILL}(Z_{n-1}^1,\ V')$ and the other is for $\mathrm{FILL}(Z_{n-1}^2,\ V')$, where $V' = V \setminus \{v_n\}$. For the latter, all arguments will be determined by the induction process. For the former, in order make the notations less cumbersome we remove V' from $\mathrm{FILL}(Z_k^1, V')$ and just write $\mathrm{FILL}(Z_k^1)$. The subscript k defines the current $|V'|$ (for a filling of a 1-dim cycle) and its actual value is $V' = V \setminus \{v_n, v_{n-1}, ...v_{k+1}\}$ for the implicitly defined pivot vertices $\{v_n, ..., v_{k+1}\}$.

Before proving the theorem we first start with an explicit expression for the degree of a vertex in Z_{n-1}^2, where Z_{n-1}^2 is the cycle generated by the call of $\mathrm{FILL}(Z_n^2,\ V \setminus \{v_n\})$ at the top level recursion. This will be used later to see how the degree of a vertex w.r.t Z_{n-i}^1 evolves in the recursion.

Claim 3.3 *Let Z_{n-1}^2 be as defined by $\mathrm{FILL}(Z_n^2, V)$ using v_n as the pivot vertex at the top recursion call. Let $u \in [n] \setminus \{v_n\}$. Then $deg(u, Z_{n-1}^2) = A(u) \oplus B(u)$ where $A(u) = deg(u, Z_n^2\ \oplus\ St(v_n, Z_n^2))$ depends only on Z_n^2 and v_n but not the implementation of FILL in the lower recursion levels. $B(u) = deg(u, \mathrm{FILL}(Z_{n-1}^1))$ depends on whether $u = v_{n-1}$ in the recursive call for $\mathrm{FILL}(Z_{n-1}^1)$ or not.*

If $u = v_{n-1}$ we have $B(u) = n - 3$.

Otherwise

$$B(u) \equiv deg(u, \mathrm{FILL}(Z_{n-2}^1)) \oplus deg(u, Lk(v_{n-1}, Z_{n-1}^1)) \pmod 2$$

Proof. Recall that by the definition of $\mathrm{FILL}(Z_n^2)$ with respect to v_n being the pivot,

$$Z_{n-1}^2 = Z_n^2\ \oplus\ St(v_n, Z_n^2)\ \oplus\ F_{n-1}^2$$

where $F_{n-1}^2 = \mathrm{FILL}(Z_{n-1}^1)$ and $Z_{n-1}^1 = Lk(v_n, Z_n^2))$.

Hence,

$$deg(u, Z_{n-1}^2) \equiv deg\left[((u, Z_n^2) \oplus St(v_n, Z_n^2))\right]\ \oplus\ deg(u, F_{n-1}^2) \equiv A(u) \oplus B(u) \pmod 2$$

Now obviously $A(u)$ depends only on Z_n, v_n but not on the implementation of F_{n-1}^2.

$B(u) \equiv deg(u, F_{n-1}^2) \equiv deg(u, \mathrm{FILL}(Z_{n-1}^1)) \pmod 2$.

Recall that using FILL recursively $\mathrm{FILL}(Z_{n-1}^1) = \mathrm{FILL}(Z_{n-2}^1) \oplus Cone(v_{n-1}, \mathrm{FILL}(Lk(v_{n-1}, Z_{n-1}^1)))$. Recall also that $Lk(v_{n-1}, Z_{n-1}^1) = Z_{n-2}^0$ is 0-dim cycle namely, an even set of vertices and hence $\mathrm{FILL}(Lk(v_{n-1}, Z_{n-1}^1))$ can be implemented to result in a 0-deficit tree T_{n-2} on $[n-2]$, whose set of odd vertices is Z_{n-2}^0.

If $u = v_{n-1}$ in the call for $\mathrm{FILL}(Z_{n-1}^1)$, $v_{n-1} \notin V(\mathrm{FILL}(Z_{n-2}^1))$ while it forms a 2-simplex with every edge of T_{n-2}, namely with $n-3$ edges. Hence the claim follows in this case.

If $u \neq v_{n-1}$ then by definition of $B(u) \equiv deg(u, \mathrm{FILL}(Z_{n-2}^1)) \oplus deg(u, Cone(v_{n-1}, T_{n-2}))$, where T_{n-2} is a tree as above. But $deg(u, Cone(v_{n-1}, T_{n-2}) = deg(u, T_{n-2}) = deg(u, Lk(v_{n-1}, Z_{n-1}^1))$ and the claim follows. ■

The core of the argument in the proof of the theorem is to analyze how the parity condition of Z_{n-1}^1 depends on Z_n^1 and the vertex v_n that is chosen to be the pivot in the top level call of FILL. It is shown next, that regardless of Z_n^2 and v_n that determine Z_{n-1}^1, the freedom in the construction of F_{n-1}^1 in the top call of FILL is enough to guarantee that Z_{n-1}^2 will be friendly.

Lemma 3.4 *Let $n \geq 7$, Z_n^2 a non empty 2-cycle and $v_n \in V(Z_n^2)$. Then there is $F_{n-1}^2 = \mathrm{FILL}(Lk(v_n, Z_n^2))$ as guaranteed by Theorem 2.7 such that Z_{n-1}^2 that is produced by the call $\mathrm{FILL}(Z_n^2)$ using F_{n-1}^2 in the top recursion level is a friendly cycle.*

Further, if $Lk(v_n, Z_n^2)$ is friendly, then there are at least two distinct such 0-deficit fillings $\mathrm{Fill}(Lk(v_n, Z_n^2))$. If $Lk(v_n, Z_n^2)$ is not friendly then there are two distinct 1-deficit fillings as above.

Proof. Let $Z_{n-1}^1 = Lk(v_n, Z_n^2)$ be the 1-cycle that is defined in the call of procedure $\mathrm{FILL}(v_n, Z_n^2)$. Let $F_{n-1}^2 = \mathrm{FILL}(Z_{n-1}^1)$ and $Z_{n-1}^2 = Z_n^2 \oplus St(v_n, Z_n^2) \oplus\ F_{n-1}^2$. To prove the claim it is enough to show that F_{n-1}^2 can be constructed so that (a) there are two vertices $x, y \in V(Z^2_{n-1})$ for which $deg(x, Z^2_{n-1}) \not\equiv deg(y, Z^2_{n-1}) \pmod 2$, (b) that F^2_{n-1} is 0-deficit or 1-deficit depending on whether Z^2_n is friendly or not, correspondingly, and (c) - that two such distinct F^2_{n-1} can be constructed for each case.

Consider the following cases:

Case 1: there are $u, u' \in V(Z^1_{n-1})$ such that $A(u) \not\equiv A(u') \pmod 2$ and $(u, u') \in Z^1_{n-1}$. Here $A(v)$ is as defined in Claim 3.3.

In that case we choose $u = v_{n-1}$ in the definition of $F^2_{n-1} = \text{FILL}(Z^1_{n-1})$, and $u' = v_{n-2}$; namely the pivot vertex in the call of $\text{FILL}(Z^1_{n-2})$ which is made in the next recursion level call in the construction of $F^2_{n-1} = \text{FILL}(Z^1_{n-1})$. We will need to show that $u' \in V(Z^1_{n-2})$ for this to be possible. Assume for now that $u' \in V(Z^1_{n-2})$.

Claim 3.3 implies that

$$deg(u, Z^2_{n-1}) \equiv A(u) \oplus B(u) \equiv \; A(u) \oplus n - 3 \pmod 2 \tag{4}$$

Also, by the same Claim,

$$deg(u', Z^2_{n-1}) \equiv A(u') \oplus B(u') \equiv \; deg(u', \text{FILL}(Z^1_{n-2})) \oplus deg(u', Lk(v_{n-1}, Z^1_{n-1})) \pmod 2 \tag{5}$$

Since $v_{n-2} = u'$, reapplying Claim 3.3 w.r.t u' and Z^1_{n-2}, we get $deg(u', \text{FILL}(Z^1_{n-2})) \equiv n - 4 \pmod 2$.

Since $(u, u') \in Z^1_{n-1}$ we have that $u' \in Lk(u, Z^1_{n-1})$ namely $deg(u', Lk(v_{n-1}, Z^1_{n-1})) \equiv 1 \pmod 2$.

Plugging the above into Equation (5) and using that $A(u) \not\equiv A(u')$, we conclude that $deg(u, Z^2_{n-1}) \not\equiv deg(u', Z^2_{n-1})$, namely that Z^2_{n-1} is friendly.

Further, Theorem 2.7 asserts that F^2_{n-1} can be made 0-deficit if Z^1_{n-1} meets the parity conditions, and of deficit 1 otherwise.

To conclude this case what is left to be shown is that we can construct Z^1_{n-2} such that $u' \in V(Z^1_{n-2})$. This is done using the relatively large freedom we have in constructing Z^1_{n-2}. The argument is formally presented in Claim B.2, Appendix B. Finally, this construction will result in one F^2_{n-1} as needed. To construct a different one with the same properties it is enough to exchange the roles of u, u' in the construction above. It is left for the reader to realize that this will result in a different F^2_{n-1} (as in particular u, u' will have different degrees with respect to Z^2_{n-1} in the two constructions).

case 2: Assuming that Case 1 does not happen then in every component of Z^1_{n-1} every two vertices x, y have $A(x) \equiv A(y) (\text{mod } 2)$.

If there are u, u' with $A(u) \equiv A(u') \pmod 2$ but $(u, u') \notin Z^1_{n-1}$, then choosing $u = v_{n-1}$ we get $B(u) = n - 3$. We show in Claim B.3 in Appendix B that Z^1_{n-2} can be constructed so that $u' \in V(Z^1_{n-2})$. Hence choosing $u' = v_{n-2}$ implies that $B(u') \equiv (n-4) + deg(u', Lk(u, Z^1_{n-1})) \equiv (n-4) \pmod{2}$ on account that $(u, u') \notin Z^1_{n-1}$. We conclude that $deg(u, Z^2_{n-1}) \not\equiv deg(u', Z^2_{n-1})$ and hence Z^2_{n-1} is friendly.

Further F^2_{n-1} is of $0/1$-deficit as needed as in the previous case. In addition, exchanging the roles of u, u' will result in a different $F^2_{n-1} = \text{Fill}(Z^1_{n-1})$ with the same desired properties, by a similar argument as in the previous case.

case 3: We are left with the case that neither case 1, nor case 2 occur. In this case either Z^1_{n-1} is the complete graph on $[n-1]$ and is monochromatic w.r.t. $A(*)$, or Z^1_{n-1} is a union of two cliques, each being monochromatic w.r.t. $A(*)$ and with different values of $A(*)$ in these two cliques. This very special case is analysed in Claim B.4 in Appendix B. It asserts that in this case too Z^1_{n-2} can be made friendly. Further two corresponding F^2_{n-1} of $0/1$-deficit are constructed as needed. ■

Proof. [of Theorem 3.2]

The proof is by induction on n. The base case is for $n \leq 7$ which we have checked by a computer program see Appendix B.5. The Theorem is in fact true for $n = 6$, but we have stated it for $n \geq 7$ so to use one computer program for every cycle (friendly or not) - see Theorem 3.5.

Let $n \geq 8$ and let Z^2_n be a friendly cycle. Let $v \in V(Z^2_n)$ for which $Z^1_{n-1} = Lk(v, Z^2_n)$ meets the parity conditions. Such v exists by the assumption of Z^2_n being friendly. Set $v = v_n$ and use the procedure FILL with v_n.

This will produce a filling $F_n^3 = \mathrm{FILL}(Z_{n-1}^2) \oplus Cone(v_n, F_{n-1}^2)$), where $F_{n-1}^2 = \mathrm{Fill}(Z_{n-1}^1)$ is as guaranteed by Lemma 3.4 to result in a friendly Z_{n-1}^2. Hence by induction $\mathrm{FILL}(Z_{n-1}^2)$ can produce two distinct 0-deficit fillings resulting in two distinct fillings for Z_n^2.

Since F_{n-1}^2 is guaranteed to be 0-deficit by Theorem 2.7, and Z_{n-1}^2 is friendly, this implies that F_n^3 is 0-deficit by induction and Lemma 2.6. ■

Theorem 3.2 immediately implies the following more general theorem.

Theorem 3.5 *Let $n \geq 7$ and Z_n^2 be a nonempty 2-cycle over $\mathbb{F}_2$ on K_n^2. Then there exist at least two acyclic filling F_n^3 of Z_n^2 of deficit that is at most 1.*

Proof. The proof is again by induction on n. For $n \leq 7$ it follows by checking finitely many possible cycles which was done by a computer program, see Appendix B.1. If Z_n^2 is friendly the assertion follows by from Theorem 3.2.

Assume that Z_n^2 is not friendly, and $n \geq 8$. Assume that for some $v \in V(Z_n^2)$, $Lk(v, Z_n^2)$ meets the parity condition. Then by Lemma 3.4 with respect to $v = v_n$, there is a 0-deficit $F_{n-1}^2 = \mathrm{FILL}(Lk(v_n, Z_n^2))$, such the resulting Z_{n-1}^2 in the top recursion level of $\mathrm{FILL}(Z_n^2)$ is friendly. Then by Theorem 3.2 there are two 0-deficit fillings F, F' each being a 0-deficit filling of Z_{n-1}^2. Using each in the top call for $\mathrm{FILL}(Z_n^2)$ together with F_{n-1}^2 we get two corresponding 0-deficit fillings for Z_n^2.

If Z_n^3 is not friendly, we pick an arbitrary $v_n \in V(Z_n^2)$ as a pivot vertex used in the top recursion level in FILL. Then Lemma 3.4 asserts that F_{n-1}^2 will be a 1-deficit filling and that Z_{n-1}^2 will be friendly. Hence Theorem 3.2 asserts at least two 0-deficit filling of Z_{n-1}^2 resulting in at least two 1-deficit filling of Z_n^2. ■

3.2 Fillings in dimension larger than 3

To prove Theorem 2.4 our intension is to use induction on the pair (d, n). The base case for $d \leq 2$ and any n is proved in Theorem 2.8 for $\mathbb{Q}$ and in Theorem 3.5 for $\mathbb{F}_2$ and $d \leq 3$. We will need a base case for every $d \geq 3$ and some small $n = n_d$. This is shown in the next claims.

Claim 3.6 *Let $n = d + 2$ and Z_n^{d-1} be a non-empty cycle over $\mathbb{Q}$. Then there are two distinct fillings for Z_n^{d-1}, each of deficit at most d.*

Proof. Every $(d-1)$ cycle $Z = Z_n^{d-1}$ can be written as $Z = \sum_{\sigma \in K_n^d} \alpha_\sigma \cdot \partial_d \sigma$, where $\alpha_\sigma \in \mathbb{Q}$ and the support of the this sum, $F = \{\sigma \mid \ \alpha_\sigma \neq 0\}$, is not empty.

Assume first $F \neq K_n^d$, namely that there is $\tau \in K_n^d \setminus F$. Note that the expression for Z defines a filling of Z supported on F. Further F is acyclic as $|F| \leq \binom{n}{d+1} - 1 = d + 2 - 1 = d + 1$ and the smallest d-cycle is of size $d + 2$.

Now to get another acyclic filling, replace for some $\sigma \in F$ the term $\partial\sigma$ with $-\sum_{\sigma' \in K_n^d,\ \sigma' \neq \sigma} \sigma'$ in the expresion for Z. Since $\partial\sigma = -\partial(\sum_{\sigma' \in K_n^d, \sigma' \neq \sigma} \sigma')$ we get again a filling F' of Z. Note that $\sigma \in F \setminus F'$, where F' is a new support after the above substitution. In particular $F' \neq F$. Hence the new sum is indeed a different filling. Further F' is acyclic by the same reasoning as above, on account of $\sigma \notin F'$ which implies that $|F| \leq d + 1$.

If $F = K_n^d$ then up to scaling we may assume that for $\sigma = (2, 3, \ldots d + 2)$, $\alpha_\sigma = 1$. In that case either for every $\tau \in K_n^d$, α_τ is identical to the coefficient of τ in $\partial_{d+1}(1, \ldots, d + 2)$. In this case $Z = \partial_d \partial_{d+1}(1, \ldots, d + 2)$ is the trivial cycle. We conclude that for some τ, α_τ is not identical to the coefficient as defined above. Now one can cancel σ from the sum representing Z by adding to the sum expressing Z the expression $-\partial_{d+1}(1, \ldots, d + 2)$ which is 0. But $-\partial_{d+1}(1, \ldots, d + 2)$ includes σ with coefficient -1 and will cancel σ from the sum. Hence, this new sum (of support at moset $d + 1$) is an acyclic filling of Z.

Alternatively getting another acyclic filling is by adding to Z the sum $-\alpha_\tau \partial_{d+1}(1, \ldots, d+2)$ which will cancel τ but will not cancel σ.

Finally, as the rank is $d+1$, the deficit of the fillings is obviously at most d. ■

A similar claim for $\mathbb{F}_2$ is as follows.

Claim 3.7 *Let $n = d+2$ and Z_n^{d-1} be a non-empty cycle. Then there are two distinct fillings for Z_n^{d-1}, each of deficit at most d.*

Proof. The proof is almost identical to that over $\mathbb{Q}$, except that the case of $F = K_n^d$ in sum expressing the cycle Z. In this later case, since all non-zero coefficients are 1, we have that $Z = Z_n^{d-1} = \sum_{\sigma \in K_n^d} \partial\sigma$. But this is just 0 (on account of $\partial\partial(1, \ldots d+2) = 0$). Namely, this case does not need any attention as Z is the trivial cycle. ■

We now prove the following stronger theorem that implies Theorem 2.4.

Theorem 3.8 *There exists a function $c : \mathbb{N} \mapsto \mathbb{N},\ d \longrightarrow c_d$ such that for every nonempty $(d-1)$-cycle Z_n^{d-1} over $\mathbb{F}_2$ or over $\mathbb{Q}$, on K_n^d, there exist at least two acyclic filling of Z_n^{d-1} each of deficit at most $c_d \cdot n^{d-3}$.*

Proof.

The proof is by induction on the pair (d, n). For $\mathbb{F}_2$, $d \leq 3$ and every n it follows from Theorem 3.5 and Theorem 2.7. For every d and small enough n it follows from Claim 3.7. Similarly, for $\mathbb{Q}$ and $d \leq 2$ it follows from Theorem 2.8. Further, for $d \geq 3$ and small enough n it follows from Claim 3.6.

The induction now is identical for both F_2 and $\mathbb{Q}$:

Let Z_n^{d-1} be a non-empty $(d-1)$-cycle for $d \geq 4$ for $\mathbb{F}_2$ or $d \geq 3$ for $\mathbb{Q}$. Let $v \in V(Z_n^{d-1})$ be arbitrary. Then applying $\mathrm{FILL}(Z_n^{d-1})$ with $v_n = v$ in the top level results in Z_{n-1}^{d-2}, the corresponding filling $F_{n-1}^{d-1} = \mathrm{FILL}(Z_{n-1}^{d-2})$ by recursion, and Z_{n-1}^{d-1}. Further, by the induction hypothesis we may assume that F_{n-1}^{d-1} is of deficit at most $c_{d-1} \cdot (n-1)^{d-4}$ (or 0 deficit if $d-1 = 2$ for $\mathbb{Q}$). Fix one such filling that results in a non-empty Z_{n-1}^{d-1} (there exists one on account of the existence of at least two distinct fillings F_{n-1}^{d-1} as above). We get by induction at least two fillings for Z_{n-1}^{d-1} each of size at most $c_d \cdot (n-1)^d$.

Then the filling that is defined by F_{n-1}^{d-2} and each of the two fillings F_{n-1}^{d-1} in the top level call of $\mathrm{FILL}(Z_n^{d-1})$ results in a filling with deficit $c_d \cdot (n-1)^{d-3} + c_{d-1} \cdot (n-1)^{d-4}$. Solving the recursion obviously results in a $c_d \cdot n^{d-3}$ deficit filling. ■

We end this section with the following conjecture that is weaker than Conjecture 2.2. It states that the procedure FILL can always be made to produce a filling with deficit that is independent on n but may depend on d.

Conjecture 3.9 *There exists a function $\alpha : \mathbb{N} \mapsto \mathbb{N},\ d \mapsto \alpha_d$ such that for every non-trivial $(d-1)$-cycle Z^{d-1} on K_n^d (w.r.t. $\mathbb{F}_2$ or $\mathbb{Q}$), $\mathrm{FILL}(Z_n^{d-1})$ can be made to produce a filling of deficit at most α_d.*

4 On the maximum size of a simple d-cycle on $[n]$

Here we use the results in Sections 2 and 3 to show the existence of large simple d-cycles. As explained in the introduction, for the very simple case of $d = 1$, Hamiltonian cycles, namely simple cycle of the maximum possible size of $r(n, 1) + 1 = n$ exist for very $n \geq 3$. For $d \geq 2$ this was open.

Let σ be a d-simplex. Recall that for an acyclic d-filling $F^{(d)}$ of the $(d-1)$-cycle $\partial\sigma$, the d-chain $F^{(d)} - \sigma$ is a simple d-cycle. Conversely, for a simple d-cycle Z and $\sigma \in Z$, $Z - \sigma$ is an an acyclic d-filling of $\partial\sigma$. Thus, Theorem 2.4, immediately imply the existence of large simple d-cycles in K_n^d over $\mathbb{F}_2$ and over $\mathbb{Q}$. This is not, however, likely to be tight.

The existence of the extreme case, that is, Hamiltonian cycles, or tighter results are of particular interest. We next sum up the consequences of Theorem 2.4 in Theorem 4.1 below.

Theorem 4.1

For $d = 2$, over $\mathbb{F}_2$, Hamiltonian 2-cycles on $[n]$ exist if and only if $n \equiv 0$ or $3 \,(mod\, 4)$. Over $\mathbb{Q}$, they exist for all $n \geq 4$ with exception of $n = 5$. In all cases there exist simple cycles of deficit ≤ 1.

For $d \geq 3$, over $\mathbb{F}_2$ as well as over $\mathbb{Q}$, there exist simple d-cycles on $[n]$ with deficit $O(n^{d-3})$. ∎

We next consider 3-dimensional cycles over $\mathbb{F}_2$. The tighter Theorem 3.5 immediately implies that there are simple 3-cycles of size $r(n, 3) = \binom{n-1}{2}$, namely of size 1-short of being Hamiltonian. This by the discussion above, and the fact that for a 3-dim simplex σ, there is a 1-deficit filling of $\partial_3\sigma$.

Note that for every $v \in Z_n^2$, $|Lk(v, \partial_3\sigma)| = 3$, hence $\partial_3\sigma$ is not friendly. Therefore Theorem 3.2 is not applicable to yield a tighter 0-deficit filling of $\partial_3\sigma$ and, in turn, a Hamiltonian 3-cycle. However, the only need of being friendly in the proof of Theorem 3.2, is to be able to choose $v = v_n$ for the top level call of $\text{FILL}(Z_n^2)$, so that the parity condition holds for the 1-dim cycle $Z_{n-1}^1 = Lk(v_n, Z_n^2)$. In our case for $Z_n^2 = \partial_3\sigma$, and as remarked above $Lk(v, Z_{n-1}^2) = 3$ for every $v \in V(Z_n^2)$. Hence whenever $\binom{n-2}{2} \equiv 1 \pmod 2$ it has the parity condition, and v could be taken so that $\text{FILL}(Z_{n-1}^1)$ is 0-deficit, resulting in a 0-deficit filling of Z_n^2. This implies, in turn, a Hamiltonian 3-cycle. We sum this in the following corollary.

Corollary 4.2 *For every $n \equiv 0, 1 \,(\text{mod} 2)$, and $n \geq 7$, there is a 3-Hamiltonian cycle in K_n^3 with respect to $\mathbb{F}_2$.*

Proof. For such n the parity condition for the 2 cycle $\partial_3\sigma$ with respect to $n - 1$ holds, and hence there is a 0-deficit filling of it resulting in a Hamiltonian cycle as explained above. ∎

In what follows we focus no $\mathbb{F}_2$ and discuss some non-trivial upper bounds for the largest simple cycles when n is relatively small with respect to d.

By a standard duality argument (see e.g., [10]), there is a size- and deficit-preserving 1-1 correspondence between the $(n - d - 2)$-hypercuts (= simple $(n - d - 2)$-cocycles), and the simple d-cycles in K_n^{n-1}. In [8], the authors discuss lower bounds on the deficit of k-hypercuts in K_n^k. In particular, it holds that:

The deficit of the largest 2-hypercut in K_n^2 is $n^2/4 - O(n)$.

For any odd k, the deficit of largest k-hypercut is at least $\binom{n-1}{k}\left(\frac{n}{(k+1)^2} - 1\right)$. (This holds for general k-cocycles as well.)

Combining these results with the above duality, and setting $k = n - d - 2$, one arrives at the following results about the deficits of d-cycles:

Claim 4.3

The deficit of the largest simple d-cycle in K_{d+4}^d is $\frac{1}{4}d^2 - O(d)$.

The deficit of any d-cycle in K_{d+k+1}^d, k odd, is at least $\binom{k+d+1}{k}\left(\frac{k+d+2}{(k+1)^2} - 1\right)$.

Corollary 4.4

For a large d and an odd $k \approx \sqrt{d} - 1$, the deficit of any d-cycle in K_{d+k+1}^d is at least $(d/e)^{0.5\sqrt{d}-O(1)}$.

5 Concluding remarks

We have shown that for every d and large enough n there is a large acyclic d-filling of any $(d - 1)$-cycle. For the case $d \leq 2$ this is completely closed (over $\mathbb{F}_2$ and over $\mathbb{Q}$). In particular, this shows the existence of very large simple d-cycles. The extremal case of Hamiltonian cycle is solved completely for $d \leq 2$. For $d = 3$ over $\mathbb{F}_2$, we have shown the existence of Hamiltonian cycles for an infinite sequence of n's. However, the existence of Hamiltonian cycles for higher dimensions is open at large. Currently we do not even see a method of approaching the problem. This poses one major open problem.

Other related open problems are proving either Conjecture 2.2 or the weaker Conjecture 3.9.

Another interesting point that follows from the discussion in this paper concerns the existence of non-collapsible trees.

A $(d-1)$-simplex τ of a pure d-dimensional simplicial complex X is called *exposed* if its degree is 1, that is, it belongs to exactly one d-simplex σ of X. An elementary d-collapse on an exposed τ as above, consists of the removal of σ and τ from X. The complex X is *collapsible to its* $(d-1)$*-skeleton* if every d-simplex of X can be removed by a sequence of elementary collapses of $(d-1)$-facets. It is easy to see that if X is collapsible to its $(d-1)$-skeleton, then $X^{(d)}$ is acyclic over any field. Is the inverse true? For $d = 1$ this is true; the fact that every acyclic graph is collapsible is identical to the fact that every non-empty acyclic graph contains a vertex of degree 1 (a leaf).

The existence of non-collapsible trees (over $\mathbb{F}_2$ and over $\mathbb{Q}$) was known, cf. []. A consequence of our results is a construction of non-collapsable d-trees for $d = 2, 3$. In fact the trees that we construct do not have any exposed $d-1$ simplex. The way to construct such trees, is to construct a Hamiltonian cycle Z, namely in which no exposed $(d-1)$-simplex exists. Further, to observe that for some d-simplex σ in it, any $\tau \in \partial\sigma$ appears with multiplicity at least 4. Hence, removing σ from Z will result a tree in which there is no exposed simplex.

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Appendix

A Case analysis for $d = 2$ over $\mathbb{Q}$ and $n \leq 6$, for the base cases of Theorem 2.8

We identify here a 1-cycle C with a weighted directed graph $C = (V, E)$ in which for every $v \in V(C)$, $\sum_{(x,v)\in C} w(x, v) - \sum_{(v,y)\in C} w(v, y) = 0$.

$n = 3$ This case is essentially empty: The unique 1-cycle is $\partial(\sigma)$ for σ being the unique 2-dim simplex. Hence σ is the required 0-deficit filling.

$n = 4$ In general all possible cycles (up to isomorphism and scaling of weights) are of the form $G_2 = \partial(123) + a \cdot \partial(234) + b \cdot \partial(124)$, for any possible $a, b \in \mathbb{R}$. It is easy to see that for $a = -1, b = 0$ one get $C_4 = \partial(1, 2, 3) - \partial(1, 2, 4) = -\partial(2, 3, 4) - \partial(1, 3, 4)$. Hence the right hand side forms a filling of size 2 which a 1-deficit filling. For every other setting of a, b (that is not isomorphic) it is easy and left for the reader to verify that C is a sum of boundary of three simplices - hence a 0-deficit filling.

$n = 5$ In this case the collection of cycles is much larger. We refer to two main cases according to whether there is a vertex v in the cycle with exactly two adjacent edges, or the case where all vertices in the cycle are adjacent to at least 3 edges.

case1: The first case in which there is a vertex v of degree 2 in C: we choose $v_5 = v$ and apply FILL. Since v has two adjacent edges, it follows that one is incoming, the other is outgoing, both with the same weight which is w.l.o.g. 1. Assume these are the directed edges $(4, 5), (5, 1)$. Then calling the FILL($(Lk(v_5, C)$, $[4]$) would find F^1_4 which is a weighted Hamiltonian path on 4 vertices from 4 to 1 carrying a weight 1 (and there are 2 such different paths). Now $Z^1_5 - St(5, Z^1_5)$ is a flow network carrying a total of 1 flow from 1 to 4. Hence (by simple flow argument) either this flow is along a simple path of length $3, 2$ or 1. Namely $Z^1_5 = C_5, C_4$ or C_3. In the first two cases F^1_4 can be taken so not to cancel this path which will result in the cycle Z^1_4 that is not C_4 and hence a 0-deficit will be constructed for it and for C.

The problematic case above is when the flow is along one path of length 1. Namely $Z_5 = C_3$. In this case, of the two possible $F^{(1)}_4$, one results in an empty cycle and the other with $Z^1_4 = C_4$ which will result in a 1-deficit filling. Hence for C_3 we end up in a 1-deficit filling. Moreover, this is best possible as it can be seen that any acyclic set on K^2_5 is a construction as described in Claim 1.1. It follows that if there were a 0-deficit filling for C_3 it would be also achieved by FILL.

We conclude that there is no such filling for C_3. We also note that there are several 1-deficit fillings.

Another subcase is when the above 1-flow from 1 to 4 is not on a simple path. It then can be split into two or more distinct paths. In that case, again, the resulting graph Z^1_4 can be made not to be C_4 resulting in a 0-deficit.

case 2: The other case that is left is where every vertex in Z^1_5 is adjacent to at least 3 edges. Assume first that there is a vertex v that is adjacent to exactly 3 edges. Assuem w.l.o.g that $v = 5$ and choose $v_5 = 5$ in FILL. HenceF^1_4 is a tree whose boundary is the three neighbours of v_5, which are w.l.o.g. $\{1, 2, 3\}$. Then any tree on $[4]$ with 4 being non-leave vertex can be made to be a suitable F^1_4. Each will result a different labeled Z^1_4. Since there are 7 such trees, with only 6 possible labeled C_4 at least one will result $Z^1_4 \neq C_4$ and the case $n = 4$ will guarantee a 0-deficit filling. (We note that none of the possible $F^{(1)}_4$ will result in an empty cycle on account that all vertices in Z^1_5 have degree at least 3).

Finally, in the case of every vertex in Z^1_5 of degree 4 makes Z^1_5 a weighted orientation of K_5. In that case $Z^0_4 = Lk(5, Z^1_5)$ is a weight $\{a, b, c, d\}$ on $[4]$ with $a + b + c + d = 0$ and F^1_4 is a weighted tree whose net weight on $[4]$ is as above. If for no proper set of $[4]$ the weights sum to 0, it is easy to see that any tree on 4 vertices can be weighted be a 0-filling of the weighted Z^0_4 as above. There are 16 such trees and only 7 forbidden configurations for Z^1_4 (the 6 labeled C_4 + the empty cycle). As each tree results in a different labeled configuration, at least one will result in a good $Z_4 \neq C_4$ for the next recursion level.

If, on the other hand Z_4^1 is the weighting $(1,-1,a,-a)$ then it can be seen that F_4^1 being any of the 4 stars can be weighted to be a filling of the above. More over, it can be seen that at least one of these possibilities will either result in Z_4^1 not being C_4. Again, by the case of $n=4$ this will result in a 0-deficit filling for C.

$n=6$ Analysis in the spirit of $n=5$ is simpler here. If there is vertex v in Z_6^1 that is adjacent to two edges we chose $v_6=v$ in FILL. Then w.l.o.g $G=Z_6^1-St(6,Z_6^1)$ is a flow network carrying a total of 1 flow from 1 to 5. Hence F_5^1 must be a Hamiltonian path from 5 to 1. Here if $|V(Z_6^1)|\le 5$ it is immediate that such path (in fact at least two paths) can be taken to result in nonempty Z_5^1 having a vertex of degree 3 (or more) and hence not C_3. It can also be verified that if $Z_6^1=C_6$ the same can be forced as well.

If the flow network defined by G is a union of two or more distinct path, again, the same holds, by the freedom we have due to the relatively large number of Hamiltonian paths.

Otherwise, if every vertex in Z_6^1 is adjacent to at least 3 edges, and there is a vertex adjacent to exactly 3 edges v we set $v_6=v$ in FILL. Assume that $V\setminus\{v\}=[5]$ and that v is adjacent to $1,2,3$, then any tree in which $4,5$ are not leaves can serve as F_5^1 (with a corresponding uniquely define weighting). As there are 30 such labeled trees and only $\binom{5}{3}=10$ labeled C_3 there is at least two trees that will produce an non-empty $Z_5^1\neq C_3$.

If all vertices in Z_6^1 have degree 4 or more, then Z_6^1 has at least 12 edges and $G=Z_6^1-St(6,Z_6^1)$ has at least 8 edges (where 6 is chosen to be the vertex of the smallest degree). But F_5^1 which is a tree on 5 vertices has 4 edges, hence added to G will result in a graph with at least 4 edges which cannot then be C_3. This ends the proof for this case.

B Claims for the proofs of Theorem 3.2

All Claims here are w.r.t 1-dim complexes, namely graphs. For a graph G we denote by

$$Odd(G)=\{v\in V(G)|\ \ deg(v,G)\equiv 1 \pmod 2\}$$

We use the following simple Claim on filling for $d=0$.

Claim B.1 *Let $O\subseteq V$ with $|O|\equiv 0\ (\mathrm{mod}2)$, $w\in O$ and $y\in V$. Then there is a 0-deficit filling of O, i.e., a tree T on V with $Odd(T)=O$ in which $St(w,T)=(y,w)$, namely the only neighbour of w in T is y.*

Proof. If $O=\{w,y\}$ then any Hamiltonian path with ends w,y is the required T. Otherwise, define $O'=(O\setminus\{w\})\oplus\{y\}$, and construct any tree T' on $[n-1]\setminus\{u,u'\}$ with $Odd(T')=O'$ (which is possible by constructing 0-deficit filling for $d=0$). Then add the edge (w,y) to T' to obtain T. ■

Claims for Case 1.

Claim B.2 *Let $u,u'\in V(Z_{n-1}^1)$ such that $(u,u')\in Z_{n-1}^1$. Let $v_{n-1}=u$ be the pivot vertex chosen in* $\mathrm{FILL}(Z_{n-1}^1)$. *There are two distinct 0-deficit filling $F_{n-2}^1,\tilde{F}_{n-2}^1$ each being* $\mathrm{Fill}(Z_{n-1}^1\setminus\{u\})$ *such that $Z_{n-2}^1=(Z_{n-1}^1\setminus\{u\})\oplus F_{n-2}^1$ contains u' in its vertex set.*

Proof.

Let $G=Z_{n-1}^1\setminus\{u\}$ be the graph on the vertex set $[n-2]$. Then $u'\in O=Odd(G)$.

If there is $y\notin\{v_n,u,u'\}$ such that $(u',y)\notin G$, then let $T=T_{n-2}$ be a tree on $[n-1]\setminus\{u\}$ as asserted by Claim B.1 w.r.t O, $w=u'$ and y. The resulting Z_{n-2}^1 that is defined by $F_{n-1}^1=T_{n-2}$ will contain the edge (u,y') and hence u' as a vertex.

The above does not happen only if in G, u' is connected to all the other $n-3$ vertices in $[n]\setminus\{v_n,u,u'\}$. Since $n-3\ge 2$ this means that it has degree at least two in G. Using the same T_{n-2} as above will result in u' being in Z_{n-2}^1. This is true as u' has at least two edges in G of which at most one can be canceled by the single edge containing u' in T. ■

Claims for Case 2.

Claim B.3 *Let $u, u' \in V(Z^1_{n-1})$ such that $(u, u') \notin Z^1_{n-1}$. Let $v_{n-1} = u$ be the pivot vertex chosen in $\mathrm{FILL}(Z^1_{n-1})$. There is a 0-deficit filling $F^1_{n-2} = \mathrm{FILL}(Z^1_{n-1} \setminus \{u\})$ such that $Z^1_{n-2} = (Z^1_{n-1} \setminus \{u\}) \oplus F^1_{n-2}$ contains u' in its vertex set.*

Proof. $F^{(1)}_{n-2}$ should be a tree T_{n-2} on $[n-2] = V \setminus \{v_{n-1}, u\}$ that has $O = Odd(T_{n-2}) = Lk(u, Z^1_{n-1})$ and such that $u' \in V(Z^1_{n-2})$ that is resulted by T_{n-2}. The construction of T^1_{n-2} in this case is simple. Construct first T' on $[n-1] \setminus \{u, u'\}$ with $Odd(T') = O$. This is possible by the 0-dim filling case. Then subdivide an edge $e = (x, y)$ of T' by replacing it with $(x, u'), (u', y)$. The resulting T_{n-2} is a tree with $Odd(T_{n-2}) = O$ regardless of the choice of e. Now, if u' is adjacent to 4 or more vertices in Z^1_{n-1} then any choice of e will result in $u' \in Z^1_{n-2}$. If u' is adjacent to exactly two neighbours a, b in Z^1_{n-1} then e can be any edge of T' except for (a, b).

Hence since $n - 4 \geq 2$, T' has at least two edges, one of them is certainly a good choice for e. ■

Claims for Case 3.

Claim B.4 *Assume that Z^1_{n-1} is composed of a disjoint union of at most two cliques, each being monochromatic with respect to $A(*)$, and of different values if it is not just one clique. Then Z^2_{n-1} can be made friendly. Moreover, two corresponding F^2_{n-1} of $0/1$-deficit as needed can be constructed.*

Proof. Suppose first that $Z^1_{n-1} = K_{n-1}$, namely it is just one clique, monochromatic with respect to $A(*)$.

Let $v = v_{n-1}$ and $G = K_{n-1} \setminus \{v\}$ the resulting clique on $[n-2]$. Then with T_{n-2} being a star centered at y, the resulting Z^1_{n-2} does not have y in its vertex set. Choosing any $u \neq y$ as $u = v_{n-2}$ as the pivot vertex in $\mathrm{FILL}(Z^1_{n-2})$ will result in a cycle Z^1_{n-3} in which $y \in V(Z^1_{n-3})$. We then choose the next pivot $v_{n-3} = y$ for $\mathrm{FILL}(Z^1_{n-1})$.

It follows (by Claim 3.3) that $deg(v_{n-1}, Z^2_{n-1}) = A(v) \oplus B(v) \equiv A(v) \oplus (n-3) \pmod 2$. On the other hand Claim 3.3 implies that $deg(y, Z^2_{n-1}) = A(v) \oplus B(y) \equiv A(v) \oplus deg(y, \mathrm{FILL}(Z^1_{n-2})) \oplus 1$ where the 1 comes from that fact that y is a neighbour of v in Z^1_{n-1}. Using Claim 3.3 again (the part on $(B(u))$), implies that $deg(y, \mathrm{FILL}(Z^1_{n-2}) = deg(y, \mathrm{FILL}(Z^1_{n-3}) \oplus 0$ where the 0 comes from the fact that y is not a neighbour of v_{n-2}. Finally, one last application of Claim 3.3 implies that $deg(y, \mathrm{FILL}(Z^1_{n-3}) = n-5$. Substituting we get $deg(y, Z^2_{n-1}) = A(v) \oplus 1 \oplus (n-5) \equiv 1 + deg(v, Z^2_{n-1})$ and we conclude that Z^2_{n-1} is friendly. We note that in all the above we assume that Z^1_{n-3} is non empty which is true since $n \geq 6$.

Assume now that Z^1_{n-1} is a disjoint union of two cliques, each monochromatic w.r.t $A(*)$. Let K_ℓ be the largest of these cliques. The situation here is very similar to the previous case: we set $v_{n-1} = v$ for an arbitrary vertex in K_ℓ. Let $y \in K_\ell \setminus \{v_{n-1}\}$. Assume we can construct a $T = T_{n-2}$ with $O = Odd(T) = V(K_\ell) \setminus \{v_{n-1}\}$ for which $(u, y) \notin Z^1_{n-2},\ u \in V(Z_{n-2})$ and $y \in K_\ell \setminus \{v, u\}$. Suppose further that choosing $v_{n-2} = u$ results in Z^1_{n-3} for which $y \in V(Z^1_{n-3})$. If such T exists then we are exactly in the situation of the previous case (w.r.t $Z_{n-3},\ y, v$) and choosing $v_{n-3} = y$ will end the proof as in that case.

To construct T' as needed, we take a star centered at y with leaves $O \setminus \{y\}$. We then subdivide an arbitrary edge of this star $e = (y, a)$ by inserting all other vertices not in K_ℓ. Namely, we replace e by a path from y to a containing all vertices not in K_ℓ. Note that $Odd(T)$ is as needed. Further, Z_{n-2} will not contain the edges (y, x) for every $x \in K_\ell \setminus \{v, y\}$. Hence choosing u to be any of these vertices x and constructing T_{n-3} as asserted in Claim B.3 w.r.t u and $u' = y$ (and n replaced by $n-1$) will result in Z^1_{n-3} that contains y in its vertex set.

We note that to apply Claim B.3 we needed $n \geq 6$ but since we have replaced n with $n-1$ we get that $n \geq 7$ is needed, which is correct by our assumptions.

Finally, the above implies a construction of one $F^2_{n-1} = \mathrm{Fill}(Z^1_{n-1})$ that is $0/1$-deficit as needed. Since in both cases Z^1_{n-1} contains a clique of size at least 3 (as $n \geq 7$), any permutation of the choices of the vertices inside one clique to play the role of u, v, y above will create an isomorphic distinct $\mathrm{Fill}(Z^2_{n-1})$ (this is since, e.g., $\mathrm{Fill}(Z^1_{n-1})$ cannot be invariant to all such permutation on account of the average degree of a pair is less than one, hence some pairs are non-existant while some pairs are, in the 1-skeleton of any acyclic $\mathrm{Fill}(Z^1_{n-1})$). ■

Theorem 3.2 for $n \leq 6$.

Claim B.5 *Theorem 3.2 is true for* $n \leq 6$.

Proof. We have checked the statement for $n \leq 7$ by running the program in Section B.1 below. It could be run for each $n \leq 7$. However, for $n = 4, 5$ the situation is simple enough to also verify manually as explained below.

For $n = 4$, which is the smallest n for which a non-empty 2-cycle exists, the unique such cycle is $Z_4^2 = \partial_4\sigma$, where $\sigma = (1, 2, 3, 4)$ is the unique 3-simplex. Then σ (of size 1) is a 0-deficit filling.

For $n = 5$ the possible non-empty 2-cycles (up to isomorphism) are: ∂K_4^3, $\partial(1, 2, 3, 5) + \partial(2, 3, 4, 5)$, and $\partial(5, 1, 2, 3) + \partial(5, 1, 2, 4) + \partial(5, 2, 3, 4)$.

The first is not friendly. It can be verified that the 1-deficit F_4^2 that is resulted by $\text{FILL}(Z_5^2) \setminus \{4\}$ will result in a non empty cycle Z_4^2. Hence by the case $n = 4$, it will have a 0-deficit filling which will result in a 1-deficit filling of Z_5^2. For the 2nd case, (which is friendly), one should, not apply FILL with the 'good' vertex, as this results in an *empty* Z_4^2. However, if one chooses the bed vertex (e.g., $v_5 = 5$) one gets $Z_4^1 = C_4$ which has two distinct 1-deficit fillings, one resulting in a non-empty Z_4^2 which by the case $n = 4$ has a 0-deficit filling. Altogether, this gives a 1-deficit filling for Z_5^2.

The same reasoning applies to the last case.

B.1 program for small n's

We have checked the case of $\mathbb{F}_2, d = 3$ and small n (for $n \leq 8$), by a $C^{+}+$ program that is available on the 2nd authors cite:

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http://cs.haifa.ac.il/˜ilan/online-papers/online-papers.html/fillings.cpp
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The program (exponential in n) runs over all possible acyclic sets on K_n^3, and for each it computes the boundary (which is a 2-cycle). In doing so, it also register for each 2-cycle how many times it was found as a boundary of a 0-deficit or a 1-deficit tree.

For $n = 7$ every cycle was found to be a boundary of at least two 1-deficit or 0-deficit trees. For $n = 8$ all cycles are boundaries of at least two distinct 0-deficit trees. ∎