The scaling of the decoherence factor of a qubit coupled to a spin chain driven across quantum critical points

Tanay Nag,[1] Uma Divakaran,[1] and Amit Dutta[1]

[1]*Department of Physics, Indian Institute of Technology Kanpur, Kanpur 208 016, India*

We study the scaling of the decoherence factor of a qubit (spin$-1/2$) using the central spin model in which the central spin (qubit) is globally coupled to a transverse XY spin chain. The aim here is to study the non-equilibrium generation of decoherence when the spin chain is driven across (along) quantum critical points (lines) and derive the scaling of the decoherence factor in terms of the driving rate and some of the exponents associated with the quantum critical points. Our studies show that the scaling of logarithm of decoherence factor is identical to that of the defect density in the final state of the spin chain following a quench across isolated quantum critical points for both linear and non-linear variations of a parameter even if the defect density may not satisfy the standard Kibble-Zurek scaling. However, one finds an interesting deviation when the spin chain is driven along a critical line. Our analytical predictions are in complete agreement with numerical results. Our study, though limited to integrable two-level systems, points to the existence of a universality in the scaling of the decoherence factor which is not necessarily identical to the scaling of the defect density.

When a quantum many-body system is slowly driven across a quantum critical point (QCP)[1] by varying a parameter in the Hamiltonian, defects are generated in the final state; this is a consequence of the diverging relaxation time close to the QCP, so that the dynamics is no longer adiabatic however slow may the variation be[2–4]. If a parameter λ of the Hamiltonian describing a $d-$dimensional system is changed linearly as $\lambda(t) = t/\tau$, $-\infty < t < \infty$, (with the QCP at $\lambda = 0$), the defect density (n) in the final state satisfies the Kibble-Zurek (KZ) scaling relation,[2–8] $n \sim \tau^{-\nu d/(\nu z + 1)}$; here, τ is the inverse rate of quenching, and ν and z are the correlation length and dynamical critical exponents, respectively, associated with the QCP.

In parallel, there are a plethora of studies which connect quantum information theory to quantum critical systems (for a review, see[7,8]). One of the major issues in this regard is the study of decoherence, namely, the loss of coherence in a quantum system due to its interaction with the environment[9]. To elucidate these studies, the central spin model (CSM) has been proposed[10]. In this model, a central spin (CS) (i.e., the qubit) has a global interaction with a quantum many body system (e.g., with all the spins of a quantum spin chain) which acts as the environment. The interaction between the qubit and the environment in fact provides two channels of time evolution of the environmental spin chain. It has been observed that the purity of the CS is given in terms of the Loschmidt echo (LE) or the decoherence factor (DF) which is the measure of the square of the overlap of the wave function evolved along the two different channels as a function of time. The LE or DF which appears in the off-diagonal term of the reduced density matrix of the qubit is minimum at the QCP signifying a maximum loss of coherence close to it which can be used as an indicator of quantum criticality[10–12].

At this point, the natural question would be what happens when the environment is driven following some protocol across a QCP. In a recent work, Damski *et al*[13], studied the decoherence of the CS by coupling it to a transverse Ising spin chain which is driven across the QCP by a linear variation of the transverse field and showed that in the limit of weak coupling, the logarithm of the non-adiabatic part of the DF (arising due to the contribution of the low-energy modes close to the critical mode and denoted by D_{non-ad}) satisfies an identical scaling to that of n, given by $\ln D_{non-ad} \sim \tau^{-1/2}$.

In present work, we consider a version of the CSM in which environmental spin chain is chosen to be an anisotropic XY spin chain which has a rich phase diagram and thereby enables us to study the scaling of the DF when the environment is quenched across different critical and multicritical points as well as gapless critical lines. The spin chain is exactly solvable using Jordan-Wigner(JW) transformations[14] and is reducible to a decoupled two-level problem in the Fourier space and hence the Schrödinger equations describing the evolution of these two levels can be analytically solved for all times. We shall, however, emphasize on an alternative method introduced in[15]. This method, valid away from the QCP, exploits the two-level nature of the reduced Hamiltonian and the exact expression for the probability of non-adiabatic transition at the final time as given by the Landau-Zener (LZ) transition formula[16]. Both the results however lead to identical scaling relations which are also verified numerically. Moreover, this alternative approach also allows us to calculate the scaling of the DF for a non-linear variation of the quenching parameter though the LZ formula is not known exactly.

Let us emphasize that our focus here is limited to only low-energy modes close to the critical mode for which the energy gap vanishes at the QCP. The high energy modes on the other hand, evolve adiabatically throughout the dynamics; though these modes contribute to the dynam-

ics of decoherence non-trivially through the fidelity factor, they do not alter the scaling relation of the DF[13].

Let us first clarify the connection between the scaling of the DF and n, that we are interested in. We shall assume weak coupling between the qubit and the environment and work within the appropriate range of time; under these circumstances, for all the quenching schemes discussed here (achieved by changing a parameter $\lambda = t/\tau$), we find the scaling relations: (i) $\ln D_{non-ad} \sim (-t^2 f(\tau))$, if QCP is at $\lambda = 0$ and (ii) $\ln D_{non-ad} \sim \{-(t - \lambda_0\tau)^2 f(\tau)\}$, if the QCP is at λ_0 (as happens for quenching through a MCP discussed below). We explore the scaling of this function $f(\tau)$ (which is found to be linear in the size of the spin chain and quadratic in system-environment coupling) with τ and address the question whether that is identical to the scaling of n. However, to eliminate t, one could further substitute $t = \lambda\tau$, to obtain the scaling $\ln D_{non-ad} \sim -\lambda^2\tau^2 f(\tau)$ (or $\sim -(\lambda-\lambda_0)^2\tau^2 f(\tau)$ for case (ii)), but it should be emphasized that the non-trivial scaling of $\ln D_{non-ad}$ with τ is provided by that of $f(\tau)$. Our studies reveal that in the cases when the environment is driven through an isolated QCP or a multicritical point (MCP), the scaling of $\ln D_{non-ad}$ (or precisely that of $f(\tau)$) is the same as that of the defect density in the final state following a quench for both linear and non-linear quenches. However, there are situations when this generic connection do not hold. For example, when the environment is driven along a critical line across the MCP, we arrive at a scaling which is significantly different from that of n.

The Hamiltonian H_E of the environment is the XY spin chain in a transverse field consisting of N spins given by[17]

$$H_E = -\sum_{i=1}^{N}[J_x\sigma_i^x\sigma_{i+1}^x + J_y\sigma_i^y\sigma_{i+1}^y + h\sigma_i^z] \quad (1)$$

which is coupled to the spin-1/2 qubit by a Hamiltonian H_{SE}. In the following, we shall define the anisotropy parameter $\gamma = J_x - J_y$, and the parameter $(J_x + J_y)$ will be set equal to unity in all cases except for the quenching through a MCP. The model is exactly solvable by JW transformation[17]; the phase diagram is shown in Fig 1.

Let us introduce the notion of DF by considering the situation in which the transverse field h is quenched as $h(t) = 1 - t/\tau$, and the qubit is coupled to the time dependent transverse field of (1) through the Hamiltonian $H_{SE} = -\delta\sum_{i=1}^{N}\sigma_i^z\sigma_S^z$, where σ_i^z is the i-th spin of the XY chain and σ_S^z represents that of the qubit. The system crosses the Ising critical points at $h = 1$ and $h = -1$ with critical mode k_c given by $k_c = \pi$, and 0, respectively. We choose the qubit to be initially (at $t \to -\infty$) in a pure state superposition $|\phi_S(t \to -\infty)\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ represent up and down states of the CS, respectively, and the environment is in the ground state $|\phi_E(t \to -\infty)\rangle = |\phi_g\rangle$. The ground state of the composite Hamiltonian $H_E + H_{SE}$, at $t \to -\infty$, is given by the
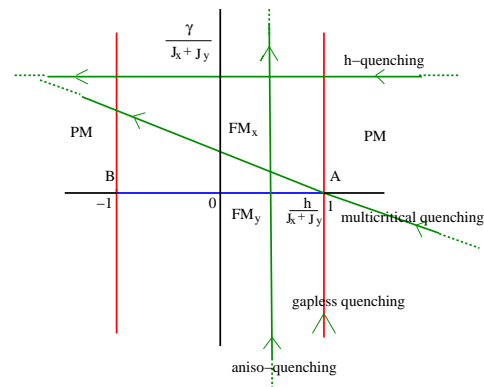

FIG. 1: (Color Online) The phase diagram of Hamiltonian (1) in the $h/(J_x + J_y) - \gamma/(J_x + J_y)$ plane. The two red vertical lines are the transverse Ising critical lines between the ferromagnetic (FM) and paramagnetic (PM) phases. The blue horizontal line for $-1 < h < 1$ corresponds to the anisotropic critical line separating the two ferromagnetic phases FM_x and FM_y, with ordering in x and y directions, respectively. Points denoted by A and B represent the two multicritical points. Green lines represent different quenching paths discussed in this paper.

direct product $|\psi(t \to -\infty)\rangle = |\phi_S(t \to -\infty)\rangle \otimes |\phi_g\rangle$. It can be shown that at a later time t, the composite wave function is given by $|\psi(t)\rangle = c_1|\uparrow\rangle \otimes |\phi_+\rangle + c_2|\downarrow\rangle \otimes |\phi_-\rangle$, where $|\phi_\pm\rangle$ are the wavefunctions evolving with the environment Hamiltonian $H_E(h\pm\delta)$ given by the Schrödinger equation $i\partial/\partial t|\phi_\pm\rangle = \hat{H}[h(t) \pm \delta]|\phi_\pm\rangle$. We therefore find that the coupling δ essentially provides two channels of evolution of the environmental wave function with the transverse field $h + \delta$ and $h - \delta$, respectively.

It is straightforward to show that the decoherence factor $D(t)$ defined as $|\langle\phi_+(t)|\phi_-(t)\rangle|^2$, is the off-diagonal element of the reduced density matrix of the qubit[13]. To evaluate $D(t)$, we rewrite the Hamiltonian (1) with modified h (due to the coupling δ) in terms of JW fermions which then can be decoupled into a sum of independent (2×2) Hamiltonians in the Fourier space[14,17]. In the basis $|0\rangle$ and $|k, -k\rangle$, which represent no quasiparticle, and quasiparticles with momentum k and $-k$, respectively, the Hamiltonian H_E can be written as

$$H_E^\pm(t) = \sum_k H_k^\pm(t), \quad \text{where,}$$

$$H_k^\pm(t) = 2\begin{pmatrix} h(t) \pm \delta + \cos k & \gamma\sin k \\ \gamma\sin k & -(h(t) \pm \delta + \cos k) \end{pmatrix}.$$

The general wave function for H_E at any instant t can be written as

$$|\phi^\pm(t)\rangle = \prod_k |\phi_k^\pm(t)\rangle = \prod_{k>0}\left[u_k^\pm(t)|0\rangle + v_k^\pm(t)|k, -k\rangle\right].$$

The coefficients $u_k^\pm$ and $v_k^\pm$ are obtained by solving the Schrödinger equation $i\partial/\partial t\left(u_k^\pm(t), v_k^\pm(t)\right)^T = H_k^\pm(t)\left(u_k^\pm(t), v_k^\pm(t)\right)^T$ where A^T represents the transpose operation of the row matrix A. Hence,

the expression of $D(t)$ is given by $\prod_k F_k(t) = \prod_k |\langle\phi_k(h(t)+\delta)|\phi_k(h(t)-\delta)\rangle|^2$, or,

$$D(t) = \exp\left[\frac{N}{2\pi}\int_0^\pi dk \ \ln F_k\right] \qquad (2)$$

where F_k can be written in terms of $u_k^\pm$ and $v_k^\pm$. We re-iterate that we shall focus in the limit of small δ and consider only the low-energy modes which show non-adiabatic behavior close to the QCP. On the other hand, the high energy modes evolve adiabatically and their overlap is close to unity. This method can be useful for exact solution as well as numerical estimation of $D(t)$.

We shall however introduce a simpler method for analytical calculations that exploits the (2×2) nature of the reduced Hamiltonian to calculate $F_k(t)$[15]. Far away from the QCP ($|h(t)|\gg 1$ ($t \to +\infty$)) i.e., after crossing both the QCPs, we can write $|\phi_k(h + \delta)\rangle = u_k|0\rangle + v_k e^{-i\Delta^+ t}|k,-k\rangle$, and $|\phi_k(h - \delta)\rangle = u_k|0\rangle + e^{-i\Delta^- t}v_k|k,-k\rangle$ where $\Delta^+ = 4\sqrt{(h+\delta+1)^2+\gamma^2\sin k^2}$ and $\Delta^- = 4\sqrt{(h-\delta+1)^2+\gamma^2\sin k^2}$ are the energy difference between the states $|0\rangle$ and $|k,-k\rangle$ when the transverse field is equal to $h + \delta$ and $h - \delta$, respectively. In writing the above expression, we make use of the fact that excitations occur only in the vicinity of QCPs. Following that the wavefunctions ($|\phi^\pm(t)\rangle$) evolve adiabatically picking up the appropriate phase factor with time. At the same time, the coefficients u_k and v_k can be found to be $|u_k|^2 = 1 - p_k$ and $|v_k|^2 = p_k$ where p_k is the Landau-Zener probability of excitations for the mode k given by $p_k = \exp(-2\pi\tau\gamma^2\sin^2 k)$[16]. Combining all these, we find

$$\begin{aligned} F_k(t) &= |\langle\phi_k(h(t)+\delta)|\phi_k(h(t)-\delta)\rangle|^2 \\ &= \left||u_k|^2 + |v_k|^2 e^{-i(\Delta^+-\Delta^-)t}\right|^2, \end{aligned} \qquad (3)$$

which can be recast in the vicinity of the quantum critical point at $h = 1$ to the form $\Delta = (\Delta^+ - \Delta^-)/2$,

$$\begin{aligned} F_k(t) &= 1 - 4p_k(1-p_k)\sin^2(\Delta t) \\ &= 1 - 4\left[e^{-2\pi\tau\gamma^2 k'^2} - e^{-4\pi\tau\gamma^2 k'^2}\right]\sin^2(4\delta t) \end{aligned} \qquad (4)$$

where $\sin k$ has been expanded near the critical modes $k = \pi$, with $k' = \pi - k$ and we have taken the limit $\delta \to 0$. The above expression is identical to that given in[13] derived via the exact solution of the Schrödinger equation.

The DF is the product of the contribution from the modes evolving adiabatically (given by fidelity) and the modes evolving non-adiabatically denoted by D_{non-ad}. The expression of $D_{non-ad}(t)$ due to the non-adiabatic dynamics of modes $k \simeq \pi$ after crossing the critical point $h = 1$ can be obtained from Eq. (2) in the following way: in the limit $\delta \to 0$ (or more precisely $(\delta t) \to 0$), one can approximate $\sin^2 4\delta t \approx 16\delta^2 t^2$ which results to

$$\begin{aligned} D_{non-ad}(t) &= \exp\frac{N}{2\pi}\int_0^\infty dk \\ &\quad \ln\left[1 - \left(e^{-2\pi\tau\gamma^2 k'^2} - e^{-4\pi\tau\gamma^2 k'^2}\right)64\delta^2 t^2\right] \end{aligned} \qquad (5)$$

where we have extended the limit of integration to ∞ since only the modes close to the critical modes contribute in the limit of large τ. Using the fact that $\ln(1-x) \sim -x$, for small x, it can be further shown that D_{non-ad} is given by

$$D_{non-ad}(t) \sim \exp\{-8(\sqrt{2}-1)N\delta^2 t^2/(\gamma\pi\sqrt{\tau})\}. \qquad (6)$$

It is worth noting that the periodicity in time as in Eq. (4) is lost and there is an exponential decay as shown in Eq. (6). This Gaussian form holds true when $t \ll 1/\delta$ and $\delta \to 0$; clearly the time range over which this is applicable increases with decreasing δ. Otherwise, a sinusoidal variation is observed. A similar expression can be obtained for the low k modes excited after crossing the $h = -1$ critical point. We find that $\ln D_{non-ad}(t)$ rather $f(\tau) = \{8(\sqrt{2}-1)N\delta^2\}/(\pi\gamma\sqrt{\tau})$ varies as $1/\sqrt{\tau}$, a scaling which is identical to that of n (with $d = \nu = z = 1$).

Above calculations can be extended to the case of the non-linear quenching of a term of the Hamiltonian, e.g., with the variation of the transverse field h given by $1 - \text{sgn}(t)(t/\tau)^\alpha$, where sgn stands for the sign function of t. Although the probability of excitation is not exactly known, casting the Schrödinger equations which describe the time evolution of the two-level systems to a dimensionless form[18], it has been argued that p_k should be a function of the dimensionless combination of k and τ given by $p_k = G(k^2\tau^{2\alpha/(\alpha+1)})$, where G is the scaling function. Considering only the contributions from the low-energy modes for large τ, one finds the scaling $D_{non-ad}(t) = \exp(-CN\delta^2 t^2/\tau^{\alpha/(\alpha+1)})$, where C is a number which also depends on α. This is again in congruence with the scaling of n for a non-linear quenching i.e., $n \sim \tau^{-\alpha/(\alpha+1)}$. This scaling has been numerically verified by directly integrating the Schrödinger equation (see discussion around Eq. (2)) and results are presented in Fig. 2(a).

In order to extract the exponent in a transparent way from the numerical data, double logarithm of D_{non-ad} is required which is numerically not possible since D is always less than unity. Hence, to calculate the exponent of τ, we introduce a modified DF, $A(\tau, t)$ given by $A(\tau, t) = -\log_{10} D_{non-ad}$. Fig. 2(a) clearly shows that $\ln A(\tau, t)$ varies linearly with $\ln \tau$ and has a slope given by $-\alpha/(\alpha + 1)$ with a fixed t, thus confirming the analytically predicted scaling relation.

If the parameter h is set equal to $2J_y$ and the interaction term J_x is quenched as t/τ, the spin chain (1) is driven across the quantum MCP A at $J_x = J_y$ or $t = J_y\tau$ (see Fig. 1); n satisfies a scaling relation[6] $n \sim \tau^{-1/6}$. This is not in agreement with the KZ prediction and has been justified by asserting the existence of quasi-critical points on the ferromagnetic side of the MCP[19]. What would happen if the environmental spin chain is driven across the MCP? Choosing appropriately the interaction H_{SE}[21], one finds $\ln D_{non-ad}(t) \sim (t - J_y\tau)^2/\tau^{1/6} \sim (J_x - J_y)\tau^{11/6}$ (see Fig. 2(b)).

Our studies have so far been limited to isolated quantum critical and multicritical points and in all cases, the
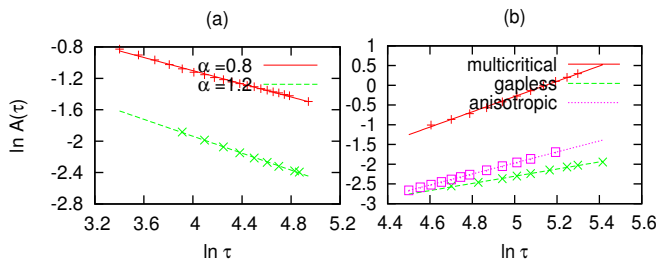
FIG. 2: (Color Online)(a) The variation of $\ln A(\tau)$ with $\ln \tau$ for two different values of α with $N = 300$ and $\delta = 0.0001$: (i) $\alpha = 0.8$ (red line) for fixed $t = 1500$ with slope -0.42 and, (ii) $\alpha = 1.2$ (green dashed line) for $t = 670$ with slope -0.54; both the slope values are very close to the predicted $-\alpha/(\alpha + 1)$. (b) The same for three different types of qubit-environment coupling with $\delta = 0.0001$ and $\alpha = 1$. (i) Line (red) corresponds to quenching across the multicritical point A in Fig. 1 by varying $J_x \sim t/\tau$ with $J_x = 7$ and $N = 500$; the slope $\simeq 1.9$ (ii) Dotted line (pink) corresponds to quenching $\gamma \sim t/\tau$ at $h = 0.5$ (across the anisotropic QCP) with $\gamma = 6.5$ and N=200, and the slope $\simeq 1.4$ (iii)Dashed line (green) is for γ quenching as t/τ along the gapless line $h = 1$ with slope $\simeq 0.9$, for $\gamma = 5$ and N=400. Analytically predicted values are 11/6, 3/2 and 1, respectively.

scaling of $\ln D_{non-ad}(t)$ (or $f(\tau)$) with τ is identical to that of n, which is not necessarily given by the traditional KZ scaling. Does this scenario hold true in general? Below we highlight a special situation where this connection between the scaling of $D_{non-ad}(t)$ and n, clearly breaks down.

Let us probe the scaling of D_{non-ad} when the parameter γ of the environment (1) is quenched as $\gamma = t/\tau$ so that the spin chain is swept across the anisotropic critical point (for $|h| < 1$) and the MCPs along the gapless Ising transition lines for $|h| = 1$ (see Fig. 1). Note that here one rewrites Eq. (1) in terms of γ with $J_x + J_y = 1$, and modifies H_{SE} to the form $H_{SE} = -(\delta/2)\sum_i(\sigma_i^x\sigma_{i+1}^x - \sigma_i^y\sigma_{i+1}^y)\sigma_S^z$. This represents a CSM in which the CS couples to the XY spin chain through the parameter γ. The coupling δ therefore provides two channels of the temporal evolution of the environmental ground state with anisotropy $\gamma + \delta$ and $\gamma - \delta$, respectively. We recall that the problem was studied in Ref.[20] from the view point of defect generation. For $|h| < 1$, $n \sim \tau^{-1/2}$, as expected from KZ theory. For the DF, one finds that $\ln D_{non-ad} \sim t^2/\tau^{1/2} \sim \gamma^2\tau^{3/2}$, which is also

numerically verified (see Fig. 2(b)). Surprise emerges for $|h| = 1$ where one finds $n \sim \tau^{-1/3}$, a scaling that can not be explained in terms of traditional KZ theory. Moreover, it was shown that $p_k = e^{-2\pi\tau(1+\cos k)^2/\sin k} \sim e^{-\pi\tau k^3/2}$ for $k \sim \pi$ when $h = 1$. Does this imply a scaling $\ln D_{non-ad}(t) \sim t^2/\tau^{1/3}$ for gapless quenching (see Fig. 1)?

To address this question, we explore $h = 1$ case in details. Using an appropriate basis[20], one can recast the reduced (2×2) Hamiltonian $H_k(t)$ to the form

$$H_k^{\pm}(t) = 2\begin{pmatrix} (\gamma \pm \delta)\sin k & h + \cos k \\ h + \cos k & -(\gamma \pm \delta)\sin k \end{pmatrix}. \tag{7}$$

Using Eq. (4) and noting that $\Delta = 4\delta k$ we find that

$$F_k = 1 - 4(e^{-\pi\tau k^3/2} - e^{-\pi\tau k^3})\sin^2(4\delta kt) \tag{8}$$

for the modes close to $k = \pi$. Assuming the limit $\delta \to 0$ and using mathematical steps identical to those employed in deriving Eq. (6) starting from Eq. (5), we once again find an exponential decay given by

$$D_{non-ad}(t) \sim \exp\{-2^{14/3}N\delta^2t^2/(3\pi\tau)\}. \tag{9}$$

We therefore find a clear deviation in the scaling of D_{non-ad} (or $f(\tau) \sim \tau^{-1}$) from $n \sim \tau^{-1/3}$. In the present case, the momentum dependence of the term $\sin^2(4\delta kt)$ in Eq. (8) renders an additional $\tau^{-2/3}$ factor resulting to a $1/\tau$ scaling of $\ln D_{non-ad}$. This clearly presents a situation where there is no direct connection between n and D_{non-ad}. Substituting $t = \gamma\tau$, one finds that $\ln D_{non-ad} \sim -(2^{14/3}N\delta^2\gamma^2\tau)/3\pi$; this is numerically verified as shown in Fig. 2(b). We note that the scaling (9) can also be reproduced analytically by solving the Schrödinger equation with equivalent reduced Hamiltonian H_k in Eq. (7).

In conclusion, we have found out the scaling of the DF (or $f(\tau)$) of a qubit coupled to a quantum spin chain which is driven across QCPs and quantum critical lines. We show that the scaling of the DF is given by the scaling of n for linear and non-linear quenching through isolated critical points. More importantly, our studies also reveal that this scenario is not universally valid.

[1] S. Sachdev, *Quantum Phase Transitions*(Cambridge University Press, Cambridge, England,1999).

[2] T. W. B. Kibble, J. Phys. A **9**, 1387 (1976), and Phys. Rep. **67**, 183 (1980). W. H. Zurek, Nature (London) **317**, 505 (1985), and Phys. Rep. **276**, 177 (1996).

[3] W. H. Zurek, *et al*, Phys. Rev. Lett. **95**, 105701 (2005).

[4] A. Polkovnikov, Phys. Rev. B **72**, 161201(R) (2005).

[5] B. Damski, Phys. Rev. Lett. **95**, 035701 (2005); J. Dziar-

maga, Phys. Rev. Lett. **95**, 245701 (2005); R. W. Cherng and L. S. Levitov, Phys. Rev. A **73**, 043614 (2006).

[6] V. Mukherjee, *et al*, Phys. Rev. B **76**, 174303 (2007).

[7] A. Polkovnikov, *et al*, Rev.Mod.Phys. **83**, 863 (2011).

[8] A. Dutta, *et al*, arXiv:1012.0653 (2010).

[9] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).

[10] H. T. Quan, *et al*, Phys. Rev. Lett. **96**, 140604 (2006).

[11] F. M. Cucchietti, *et al*, Phys. Rev. A **75**, 032337 (2007); D.

Rossini, *et al*, Phys. Rev. A **75**, 032333 (2007); J. Zhang, *et al*, Phys. Rev. A **79**, 012305 (2009).

[12] Lorenzo C Venuti and P. Zanardi, Phys. Rev. A **81**, 022113 (2010); Lorenzo C Venuti, *et al*, Phys. Rev. Lett. **107**, 010403 (2011).

[13] B. Damski, *et al*, Phys. Rev. A **83**, 062104 (2011).

[14] E. Lieb, *et al*, Ann. Phys.(N.Y.) **16** 37004 (1961).

[15] F. Pollmann,*et al*, Phys. Rev. E **81**,020101 (R) (2010).

[16] C. Zener, Proc. Roy. Soc. London Ser A **137**, 696 (1932); L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-relativistic Theory*, 2nd ed. (Pergamon Press, Oxford, 1965).

[17] J.E. Bunder and R. H. McKenzie, Phys. Rev. B., **60**, 344, (1999).

[18] Diptiman Sen, *et al*, Phys. Rev. Lett. **101**, 016806 (2008).

[19] U. Divakaran, *et al*, J. Stat. Mech. P02007 (2009); S. Deng, *et al*, Phys. Rev. B **80**, 241109 (R) (2009);V. Mukherjee and A. Dutta, EPL **92**, 37004 (2010).

[20] U. Divakaran, *et al*, Phys. Rev. B **78**, 144301 (2008); S. Deng, *et al*, EPL, **84**, 67008 (2008).

[21] S. Sharma, *et al*, Eur. Phys. Jour. B **85**:143 (2012).